Persistence-Dependent Optimal Policy Rules

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Abstract

A policy target (for example inflation) may depend on the different persistent components of exogenous cost-push shocks. These shocks may affect energy or imported prices. The larger the persistence of these exogenous shocks, the larger the welfare losses and the larger the response of policy instrument to this exogenous shock in a negative-feedback rule. This decreases the sensitivity of the policy target to the cost-push shock, close to zero, for very high persistence.

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1 Introduction

Should macroeconomic policy respond to exogenous persistent shocks, such as cost-push shocks, sometimes driven by energy prices? Barnett *et al.* (2019) neatly summarizes this issue as follows:

> "Considerable energy price hikes produce a stark distinction between headline and core inflation rates. Headline inflation measurement is similar to that from the conventional consumer price index (CPI), while core inflation removes some of the CPI's most volatile components, such as energy and food prices. Which inflation rate is the appropriate target for monetary policy? Opinions among researchers and policymakers are divided on this subject. Energy price hikes can produce a phenomenon resembling bifurcation in New Keynesian model dynamics, thereby altering the optimal policy regime. On the general subject of bifurcation of New Keynesian models, see Barnett and Duzhak (2010)... In practice, when framing the objectives and conducting monetary policy, both the Bank of England and the European Central Bank target the headline inflation rate. However, other central banks, including the U.S. Federal Reserve, pay more attention to the core inflation rate, at least in describing their operational decisions."

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A theoretical view is that the standard New-Keynesian model predicts that optimal policy should respond to price fluctuations in the sectors with sticky prices, but not in the sectors with flexible prices (such as energy prices), *regardless of the persistence* of the cost-push shock. The argument is that sticky prices lead to inefficient price dispersions. But the standard New-Keynesian model always includes the *sensitivity* of inflation to *persistent* cost-push shocks, which *also* matters. The argument is that the dependence of inflation to exogenous persistent shocks causes lasting inefficient delays for the adjustment of inflation to the long-run inflation target.

Hence, Ramsey policy (cf. Sorger (2015)) *optimally decreases the sensitivity of inflation to a persistent cost-push shock*. To this end, the policy instrument responds to a measure of the exogenous cost-push shock. This paper demonstrates that Gali (2015) Ramsey optimal policy with the new-Keynesian Phillips curve transmission mechanism behaves this way.

Changing Gali's (2015) calibration of the persistence from an autocorrelation of 0.8 to a unit root corresponds, for example, to an imported inflation trend for energy prices resembling a bifurcation in new-Keynesian model dynamics, as mentioned by Barnett *et al.* (2019)). This leads to an optimal larger response of the policy instrument to the cost push shock in order that the sensitivity of inflation to the cost-push shock is nearly zero. For example, inflation converges in four quarters to its long run inflation target. Inflation avoided a bifurcation towards trend inflation. But the policy instrument remains highly persistent for the time the highly persistent cost-push shock remains. After four quarters, according to optimal policy, the policy instrument remains proportional to the highly persistent cost-push shock.

To our knowledge, this optimal negative feedback mechanism where the policy instrument response decreases the sensitivity of inflation to the cost-push shock has never been explicitly put forward in standard New-Keynesian theory. Using linear substitutions concealed the negative-feedback mechanism of decreasing the sensitivity of inflation to the cost-push shock using the response of the policy instrument to the cost-push shock. For example, one can do linear substitutions in order to find another representation of the optimal policy rule so that the response of the policy instrument to the cost-push shock is substituted by a lag of the policy instrument. This is exactly Gali (2015) representation of the optimal policy rule of Ramsey optimal policy for the new-Keynesian Phillips curve transmission mechanism. Chatelain and Ralf (2019) demonstrates the observational equivalence of impulse response functions and the equivalence of the systems of equation using linear substitutions between Gali (2015) representation of the policy rule and the one where the policy instrument responds to inflation and to the cost-push shock.

Moreover, Ramsey optimal policy theoretically grounds Ashley, Tsang and Verbrugge's (2020) empirical evidence that the Fed funds rate response increases with the persistence of shocks. They estimated six different parameters in the Taylor rule depending on the low, medium or high persistence components of output gap or of inflation shocks. Ashley and Tsang (2013) also extracted three components of oil prices changes with low, medium and high persistence. These components may enter into exogenous cost-push shocks driving inflation.

For example, if a cartel of foreign countries controlling oil price reaches an agreement for increasing or decreasing oil price, and if this agreement appears fragile in the future, because of the lack of synchronicity of the business cycles of these countries, the central bankers may decide that their policy instruments should respond weakly to an expected short lived shock on oil price with a low persistence.

By contrast, if a rising trend of oil price is credibly expected for the next two or three years by central bankers, the central bankers of small open economies depending on imported oil price may decide that the central bank policy instrument should strongly respond to this highly persistent oil price shock. This avoids the transmission of trend inflation to home inflation for the next two to three years. Optimal policy can be implemented as follows. For example, if there is a precise forecast for persistent trend inflation for energy prices in the next three years, the policy funds rate should increase if there is a high sensitivy of headline inflation to energy price.

For distinct persistence components of cost-push shocks, this paper neatly highlights these three theoretical results with closed form solutions and with Gali's (2015) calibration for Ramsey optimal policy for the new-Keynesian Phillips curve transmission mechanism.

Firstly, the optimal response of the policy instrument to exogenous persistent shocks increases with the auto-correlation of shocks. Although the central bank cannot control the persistence of exogenous shocks, it can control the sensitivity of inflation to these shocks with its policy rule response and alter the overall persistence of inflation.

Secondly, optimal policy will reduce this sensitivity to zero - that is, the policy response will make inflation quite insensitive to these shocks - as the persistence of the shocks tends toward one. This is one explanation among others for the observed smaller order of the dynamics of the policy target (a smaller number of lags) than the order of the dynamics (the number of lags) of the policy instrument.

Thirdly, a non-zero weight in the loss function of the variance of the policy instruments (which respond to persistent exogenous shocks) implies that welfare depends on the volatility of persistent exogenous shocks. This occurs even if the loss function has sets a zero weight on the volatility of persistent exogenous shocks.

Following these results, a section is devoted to the interpretation of the larger persistence of the Fed funds rate in Taylor rule estimations with respect to the persistence of US inflation since 1982 highlighted by Fuhrer (1990).

Svensson (2003) argued that optimal policy brings more insights on central bank behavior than assuming simple rules with given constant parameters. In particular, this is true when central bankers face a number of low, medium and high persistent shocks, as currently estimated in dynamic stochastic general equilibrium models. These results are promising venue for further analysis of optimal monetary policy for various models of the transmission mechanism including persistent shocks.

2 Persistence-Dependent Optimal Policy Rule

The policy maker's optimal linear quadratic policy program in the general case is:

$$\max_{x_t} -\frac{1}{2}E_0 \sum_{t=0}^{t=+\infty} \beta^t \left[Q_{\pi^2}\pi_t^2 + 2Q_{\pi z}\pi_t z_t + Q_{z^2}z_t^2 + Rx_t^2\right] \tag{1}$$

with $R > 0$, $\mathbf{Q} \geq 0$, $0 < \beta \leq 1$, subject to:

$$\begin{pmatrix} \pi_{t+1} \\ z_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} A & A_{\pi z} \\ 0 & \rho \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix} + \underbrace{\begin{pmatrix} B \\ B_{zx} = 0 \end{pmatrix}}_{\mathbf{B}} x_t + \begin{pmatrix} 0 \\ \eta_t \end{pmatrix} \tag{2}$$

with $A > 0$, $A_{\pi z} \neq 0$, $B \neq 0$ and $0 < \rho \leq 1$.

where E_t denotes the expectation operator, π_t denotes the rate of inflation between periods $t - 1$ and t in deviation of its optimal long run equilibrium value. The policy instrument is x_t, which may represent (as in Gali (2015) the welfare-relevant output gap, i.e. the deviation between (log) output and its efficient level.

The policy maker preferences can be derived from specific micro-foundations related to specific distortions. We apply our results using Gali's (2015) endogenous welfare policy weight for a new-Keynesian Phillips curve. In this case, preferences are such that the cost of changing the policy instruments depends on the ratio of the slope of the new-Keynesian Phillips curve (κ) denoted divided by the price elasticity of intermediate goods (ε): $R = \frac{\kappa}{\varepsilon}$ for a weight on the volatility of inflation equal to one: $Q_{\pi^2} = 1$. As usual, the weights of the exogenous autoregressive cost-push shocks and of its covariance with inflation are equal to zero: $Q_{\pi z} = Q_{z^2} = 0$. As demonstrated later, these preferences *do not imply* that the value function, once optimal policy has been designed, does not depend on the volatility of the shock.

In the more general case, the policy maker preferences includes a discount factor β and weights $\mathbf{Q}$ on the variance and covariance of inflation and of the persistent cost-push shock and R on the variance of the policy instrument. $\mathbf{Q}$ is a positive matrix. It is usually assumed that the weights related to the cost-push shock are zero: $Q_{\pi z} = Q_{z^2} = 0$. Furthermore, a zero weight on inflation $Q_{\pi^2} = 0$ may be assumed if the central bank with maximal inertia is extremely reluctant to change its policy instrument. The policy instrument smoothing parameter R is strictly positive. In order to derive closed-form solutions, we consider only one policy target and only one policy instrument, one of the two weights Q_{π^2} or R can be set to the value one. The general case where policy targets, persistent shocks and policy instruments are vectors can be numerically solved using Chatelain and Ralf algorithm (2019, 2020).

Following Chatelain and Ralf (2019, 2020) algorithm, the optimal proportional negative-feedback policy rule is a linear function of inflation and output gap with reduced form rule sensitivities: F_π for the response of the policy instrument to inflation, F_z for the response of the policy instrument to persistent shocks. These optimal sensitivities are non-linear functions of the parameters of the preferences of the policy maker $(\beta, \mathbf{Q}, \mathbf{R})$ and of the transmission mechanism in the pair $(\mathbf{A}, \mathbf{B})$:

$$x_t = F_\pi \pi_t + F_z z_t \tag{3}$$

A persistent exogenous cost-push shock z_t may correspond to oil, energy or imported inflation due to foreign supply or demand shocks, or to exogenous exchange shocks or to exogenous shocks on the price of labour or other inputs. The policy maker's instrument cannot change the persistence of this shock ($B_{zx} = 0$). This cost-push shock follows an autoregressive process of order one (AR(1)) with identically and independently normally distributed white noise disturbances η_t of variance σ_η^2, with an autocorrelation coefficient $0 < \rho \leq 1$. The full dynamic system of inflation and of the cost-push shock is of order two.

Monetary policy, by construction, alters the dynamics of inflation using a negative-feedback mechanism. This is displayed clearly in our framework: in equilibrium, the evolution of inflation is governed by the new-Keynesian Phillips curve (2) after substituting in the negative-feedback policy function, Equation (3). Upon performing this substitution, we find the closed-loop equation (4):

$$\pi_{t+1} = (A + BF_\pi)\,\pi_t + (A_{\pi z} + BF_z)\,z_t \tag{4}$$

We define post-policy "intrinsic" inflation persistence as $\lambda = A + BF_\pi$. We define "post-policy sensitivity to the cost-push shock" as $\delta = A_{\pi z} + BF_z$. As a point broader than our paper, this result is important to remember when interpreting reduced-form parameters estimates of the pair (λ, δ) of accelerationist, new-Keynesian Phillips curves or inflation equation including a cost-push shock variable, such as oil price. These reduced-form estimates face the Lucas (1976) critique (Chatelain and Ralf (2020b)).

We determine analytically the closed-form sensitivity $F_z(\rho)$ of the policy instrument to the persistence ρ of the cost-push shock. Using comparative statics, we demonstrate that it increases with the persistence of the cost-push shock $0 < \rho \leq 1$.

In a second step, the cost push shock is the sum of several components ordered by their persistence up to a unit root. Their persistence is measured by auto-correlation parameters, for a diagonal vector auto-regressive matrix of order one:

$$z_t = \sum_{t=1}^{t=T} z_{i,t}, \text{ with } 0 < \rho_1 < ... < \rho_T \leq 1 \tag{5}$$

There is a larger response of the policy instrument to the more persistent component of the cost push shock, with rule parameters following the same order than the persistence parameters:

$$x_t = F_\pi \pi_t + \sum_{t=1}^{t=T} F_z(\rho_i)\, z_{i,t} \text{ with } F_z(\rho_1) < ... < F_z(\rho_T) \tag{6}$$

For a non-diagonal vector auto-regressive matrix of the cost-push shocks components, the components of the policy rule $F_z(\rho_i)$ can be numerically obtained adapting the SCILAB code in the appendix to the dimension of this matrix. This code is based on Chatelain and Ralf (2019) algorithm.

Because of Simon (1956) certainty-equivalence result for linear quadratic dynamic optimization, the optimal response of the policy instrument is not sensitive to the magnitude of transitory shocks $\eta_{i,t}$ measured by its standard error $\sigma^2_{\eta_i}$. Each of the components $z_{i,t}$ may have distinct standard errors of their transitory shocks $\sigma^2_{\eta_i}$ without changing the sensitivity of the policy instrument to each component $F_z(\rho_i)$. Simon (1956) is a first result of persistence-dependent optimal policy rule, because it states as the policy maker should not respond to transitory shocks but only to persistent shocks.

Our results do not depend on specific relations between the parameters $(\beta, A, A_{\pi z}, B)$ of the policy transmission mechanism. They hold for backward-looking models assuming inflation is predetermined or for Ramsey optimal policy where inflation is non-predetermined (jump) variable and optimally anchored at its initial date by the policy maker setting an initial optimal value of its policy instrument (Gali (2015), Sorger

(2015)).

The persistence-dependent results can be extended to multiple policy targets, multiple persistent shocks and multiple policy instruments using Chatelain and Ralf (2019, 2020) algorithm. In this case, only numerical values are available. Closed form solutions can only be found for models of order two with single policy target, single persistent shock and single policy instrument.

To illustrate the usefulness of our framework, we apply our results to Gali's (2015) new-Keynesian Phillips curve (NKPC) transmission mechanism:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + z_t \Leftrightarrow E_t \pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta} x_t - \frac{1}{\beta} z_t, \tag{7}$$

with parameters given in table 1:

Table 1: New-Keynesian Phillips curves parameters and welfare preferences, $\beta = 0.99$**,** $\kappa = 0.1275$, $\varepsilon = 6$.

Parameters	A	B	$A_{\pi z}$	Q_{π^2}	$Q_{\pi z}$	Q_{z^2}	R	ρ
NKPC	$\frac{1}{\beta} \geq 1$	$-\frac{\kappa}{\beta} < 0$	$-\frac{1}{\beta} = -A$	1	0	0	$\frac{\kappa}{\varepsilon}$	ρ
Calibration	$\frac{1}{0.99}$	$-\frac{0.1275}{0.99}$	$-\frac{1}{0.99}$	1	0	0	$\frac{0.1275}{6} = 2.125\%$	0.8

Gali's (2015) example has a large sensitivity (close to one in absolute value) between inflation and cost-push shock: $A_{\pi z} = \frac{-1}{\beta} = \frac{-1}{0.99}$. One percent change of cost-push shock leads to one percent change of inflation ($A_{\pi z} = -\frac{1}{\beta} = -1.01$). Gali's (2015) calibration uses welfare computation for the weight on the variance of the policy instrument that is extremely small ($R/Q_{\pi^2} = 2\%$). This implies a policymaker who is nearly a strict inflation targeter, with little concern about the volatility of the output gap. As we will see, these parameters imply very large absolute magnitudes of the optimal policy parameters consistent to an allowed large variance of the policy instrument.

The results of optimal policy are summarized in the following propositions. Proofs are in the appendix.

Proposition 1 *Optimal "intrinsic" inflation persistence is controllable by the policy maker and equal to* $\lambda = A + BF_\pi$. *The optimal response* F_π *of the policy instrument to policy target does not depend on the exogenous persistence* (ρ) *of cost-push shock nor on the sensitivity of inflation to these cost-push variables* ($A_{\pi z}$)*:*

$$0 < \lambda = \frac{1}{2}\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R}\frac{B^2}{A} - \sqrt{\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R}\frac{B^2}{A}\right)^2 - \frac{4}{\beta}} \right) \tag{8}$$

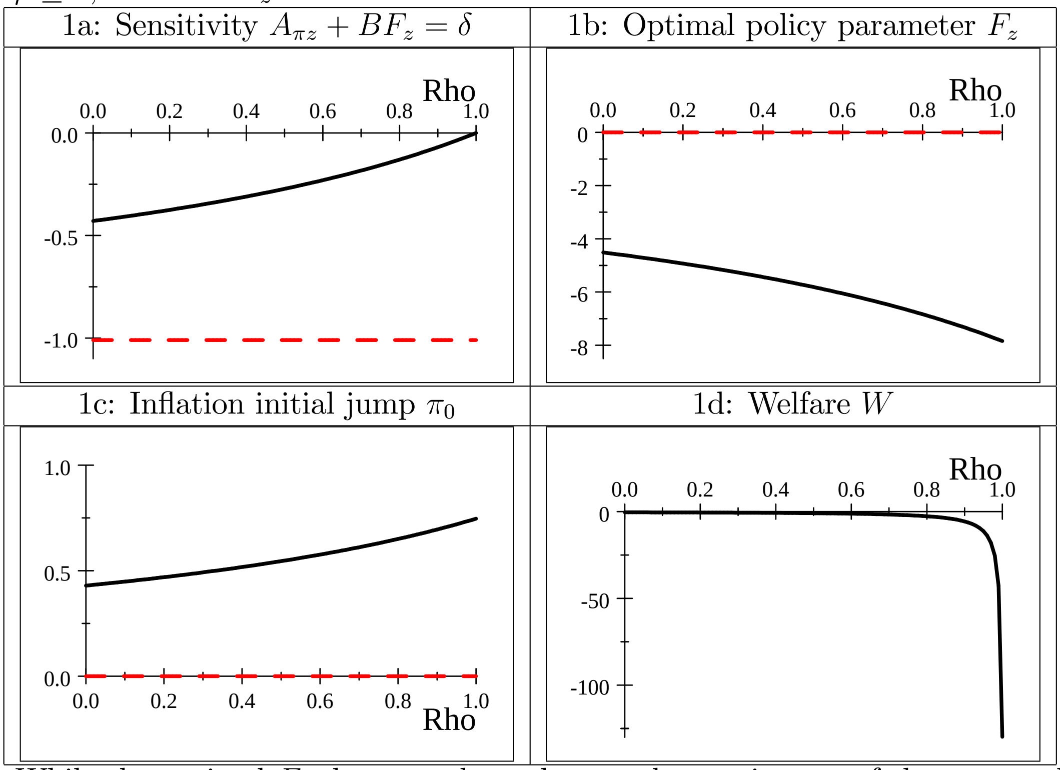
$$0 < \lambda \leq \min\left(A, \frac{1}{\beta A}\right) \text{ upper bound obtained for } Q_{\pi^2} = 0$$

$$F_\pi = \frac{\lambda - A}{B} \text{ and } BF_\pi < 0 \text{ so that } \lambda = A + BF \leq A.$$

Example 1 *For Gali's (2015) calibration, because the welfare weight on the variance of the policy instrument in the loss function is implausibly low* ($R/Q_{\pi^2} = 2\%$)*, this implies a very low "intrinsic" persistence of inflation* ($\lambda = 0.429$) *due to an implausibly large negative-feedback response of the policy instrument* ($F_\pi = 4.51$) *to deviations of inflation from its long run target.*

Using the Gali (2015) calibration (aside from ρ), Figure 1 illustrates some of the findings in Propositions $2 - 6$. We compare optimal policy to a policy function which does not react to cost-push shocks ($F_z = 0$). Figures 1a-1d highlights the dependence on the persistence of shocks (ρ) of the response of the policy instrument F_z to persistent shocks (Figure 1b), of the resulting sensitivity of the policy target to the cost-push shock $\delta = A_{\pi z} + BF_z$ (Figure 1a), of the initial jump of non-predetermined inflation π_0 (Figure 1c) and of welfare W (Figure 1d). Welfare is a particular case of Chatelain and Ralf (2020).

Figure 1: Key parameters functions of cost-push shock persistence: continuous line $0 < \rho \leq 1$, dash line $F_z = 0$.



While the optimal F_π does not depend upon the persistence of the cost-push shock process, the optimal F_z, the response to the cost-push shock itself, does depend on the persistence ρ of the shock process. Optimal policy weighs against the cost-push shock so as to diminish its impact. As the shock process becomes more persistent, the policy parameter weighs against it more intensely, so that sensitivity (δ) is decreasing in this persistence ρ.

Proposition 2 *The absolute value of the sensitivity of the policy target to the cost-push shock after policy rule response is below the absolute value of the sensitivity before policy intervention $|A_{\pi z} + BF_z| < |A_{\pi z}|$. It is a non-linear decreasing function of the persistence of the of the cost-push shock measured by ρ. It is an affine function of the sensitivity $A_{\pi z}$ of the policy target to the persistent shock when the policy instrument does not respond to cost-push shock. We denote $\delta = A_{\pi z} + BF_z$*

$$\delta = A_{\pi z} + BF_z = \frac{\lambda}{A} \frac{1}{1 - \lambda\beta\rho} \left(A_{\pi z} (1 - A\beta\rho) - Q_{\pi z} \frac{\beta B^2}{R} \rho \right)$$

One has:

$$|A_{\pi z} + BF_z| < |A_{\pi z}| \text{ because } \frac{BF_\pi}{A} = \frac{\lambda - A}{A} < 0$$

When cost-push shock tends to zero persistence, the sensitivity tends a lower sensitivity *than the one obtained for $F_z = 0$.*

$$\lim_{\rho \to 0} |A_{\pi z} + BF_z(\rho)| = \left| A_{\pi z} \frac{\lambda}{A} \right| \leq |A_{\pi z}|$$

When cost-push shock persistence tends to a unit root, the absolute value of the sensitivity of the policy instrument to the cost-push shock reaches its lowest value:

$$\lim_{\rho \to 1} (A_{\pi z} + BF_z) = \frac{\lambda}{A} \frac{1}{1 - \beta\lambda} \left(A_{\pi z} (1 - A\beta) - Q_{\pi z} \frac{\beta B^2}{R} \right)$$

Because $\lambda \neq 0$ and $\beta \neq 0$, this lowest value is equal to zero for $A = 1/\beta$ (which is a property of the new-Keynesian Phillips curve) and for a zero weight on the covariance of inflation and cost push shock $Q_{\pi z} = 0$ in the loss function.

Example 2 *As seen in figure 1, for Gali (2015) preference $Q_{\pi z} = 0$, $A = 1/\beta$ and persistence $\rho = 0.8$ of the cost-push shock, because the welfare weight on the variance of the policy instrument in the loss function is extremely low (2%), the large response of the policy instrument to the cost-push shock implies that the sensitivity of the inflation to the cost push shock is 13% of what it would have been if the policy instrument would not have responded to the persistent cost-push shock ($F_z = 0$):*

$$A_{\pi z} + BF_z = \frac{-\lambda(1-\rho)}{1 - \beta\lambda\rho} = \frac{-0.4291604 \cdot (1-\rho)}{1 - 0.4291604 \cdot 0.99 \cdot \rho} = -0.13$$

$$\lim_{\rho \to 1} (A_{\pi z} + BF_z) = 0 \text{ because } A = \frac{1}{\beta} \text{ and } Q_{\pi z} = 0.$$

$$\lim_{\rho \to 0} |A_{\pi z} + BF_z| = |-\lambda| = 0.429 < \left| -\frac{1}{\beta} \right| = 1.01$$

Proposition 3 *Inflation overall persistence and impulse response function firstly depends on its "intrinsic" persistence (controllable root $\lambda = A + BF_\pi$) and secondly depends on the non-controllable persistence ρ of the cost push shock, which is itself attenuated by the policy instrument response F_z to the cost-push shock decreasing the sensitivity of inflation to the cost-push shock $A_{\pi z} + BF_z$.*

$$E_t \pi_t \left(\underset{+}{A_{\pi z}}, \underset{+}{\rho} \right) = \lambda^t \pi_0 + \frac{\rho^t - \lambda^t}{\rho - \lambda} \delta z_0 \text{ if } \rho \neq \lambda$$

The initial value z_0 is equal to the initial value of the disturbance of the shock $z_0 = \eta_0$. Under the condition $A = \frac{1}{\beta}$, if the cost-push shock autocorrelation tends to one, the order of the dynamics of the policy target is reduced to one (single eigenvalue λ):

$$\lim_{\rho \to 1} E_t \pi_t = \lambda^t \pi_0 \text{ if } A = \frac{1}{\beta} \text{ and } Q_{\pi z} = 0 \text{ so that } \delta \to 0$$

Example 3 *For Gali's (2015) calibration, the impulse response function of inflation is lower when the policy rule responds to the cost-push shock ($F_z^* \neq 0$) than when the policy rule does not respond to the cost-push shock ($F_z = 0$). This is because the sensitivity of* *inflation to cost push shock is* 13% *of* $-\frac{1}{\beta}=-1.01$*:*

$$E_t\pi_t = 0.429^t\pi_0 + \frac{0.8^t - 0.429^t}{0.8 - 0.429}(-0.13)\, z_0$$
$$\lim_{\rho\to 1} E_t\pi_t = 0.429^t\pi_0$$

In the limit case of unit root persistence for the cost push shock ($\rho \to 1$) *such as a trend in oil price, the sensitivity of inflation to the cost push shock* ($A_{\pi z}+BF_z$) *tends to zero, so that inflation is an order one process (with single root* λ*) instead of an order two process depending on two roots (*λ *and* ρ *of the cost-push shock).*

Figures 2 show impulse response functions for inflation, for its policy instrument and for the cost-push shock for 12 quarters following a cost-push shock of one unit at the initial data. For figure 2a, persistence is a quasi-unit root $\rho = 0.99$ for the cost-push shock. Such a persistent shock may be related to a exogenous trend that can be imported to home inflation. Inflation is nearly back to equilibrium in four quarters. Afterwards, the policy instrument is proportional to the shock $x_t = F_\pi 0 + F_z(0.99)\, z_t$. It remains for a long time at a high level because z_t remains large. The proportion $F_z(0.99) = -7.79$ is constant and large in order to decrease to near-zero the dependence of inflation on the highly persistent cost-push shock. Such a large value is obtained because the weight on the volatility of the policy instrument is extremely low in Gali (2015) calibration.

For figure 2b, persistence is $\rho = 0.8$. The impulse response functions are exactly the ones presented in Gali (2015), chapter 5. Inflation impulse response is nearly the same than for $\rho = 0.99$. Inflation is nearly back to equilibrium in four quarters. As in figure 2a, Inflation is nearly back to equilibrium in four quarters. Afterwards, the policy instrument is proportional to the shock $x_t = F_\pi 0 + F_z(0.8)\, z_t$. Its absolute magnitude decreases much faster in figure 2a because z_t decreases much faster towards equilibrium. The proportion $F_z(0.8) = -6.86$ is constant and large but lower than $F_z(0.99)$. It decreases the dependence of inflation on the highly persistent cost-push shock. This large value for F_z is obtained because the weight on the volatility of the policy instrument in the quadratic loss function is extremely low in Gali (2015) calibration.

INSERT FIGURE 2a AND 2b HERE

Proposition 4 establishes that the magnitude of F_z, the policy response to the cost-push shock, increases with the persistence ρ. It also establishes the (stringent) conditions under which optimal policy does not respond to the cost-push shock. In particular, there is no response to the cost-push shock only if the persistence of the cost-push shock is zero, or, if this persistence is strictly positive, only if the policy maker does not weight the volatility of the policy target in the loss function ($Q_\pi = 0$) and if the inflation without policy intervention is stationary ($A < 1$).

Proposition 4 *The policy instrument has a persistence-dependent rule parameter* F_z *increasing in absolute value with the persistence* ρ *of the cost-push shock. It increases in absolute value with the sensitivity* $A_{\pi z}$ *of the policy target to the cost-push shock:*

$$F_z\left(\underset{+}{A_{\pi z}}, \underset{+}{\rho}\right) = \frac{\delta - A_{\pi z}}{B} = F_z = \frac{A_{\pi z}}{A}\frac{1}{1-\beta\lambda\rho}\frac{\lambda - A}{B} - \frac{Q_{\pi z}}{A}\frac{B}{R}\frac{\beta\lambda\rho}{1-\beta\lambda\rho}$$

The limit of the function $F_z(\rho)$ *is not necessarily zero if persistence tends to zero, although when* $\rho = 0$*, the policy rule does not respond to transitory shocks:* $F_z = 0$

(Simon (1956)):

$$\lim_{\rho\to 0}|F_z(\rho)| = \left|\frac{A_{\pi z}}{A}\frac{\lambda - A}{B}\right| > 0 = F_z \text{ if } \rho = 0, \text{ otherwise:}$$
$$\lim_{\rho\to 0}|F_z(\rho)| = 0 \text{ if } \rho > 0,\ \lambda = A < 1 \text{ and if } Q_{\pi^2} = 0$$

The lack of response of the policy instrument when the persistence of the shock tends to zero is obtained only in an irrelevant case where welfare and the policy maker do not weight the volatility of the policy target in the loss function $Q_{\pi^2} = 0$ and if inflation is stationary for a fixed setting of the policy instrument at its long run equilibrium value ($\lambda = A < 1$).

Example 4 *As seen in figure 1, for Gali (2015) persistence $\rho = 0.8$ of the cost-push shock, because the welfare weight on the variance of the policy instrument in the loss function is extremely low (2%), this allows large variations of the policy instrument with a large response of the policy instrument to the cost-push shock (-6.86):*

$$F_z = \frac{1}{\kappa}\left(-\frac{1-\beta\lambda}{1-\beta\lambda\rho}\right) = \frac{1}{0.1275}\left(-\frac{1-0.4291604\cdot 0.99}{1-0.4291604\cdot 0.99\cdot\rho}\right) = -6.86$$
$$\lim_{\rho\to 0}|F_z(\rho)| = |F_\pi| = 4.52 > |F_z(\rho=0)| = 0 \text{ and } \lim_{\rho\to 1}|F_z(\rho)| = 7.84$$

Proposition 5 *If inflation π_0 is non-predetermined, in Ramsey optimal policy, it is optimally anchored on the policy instrument which is itself anchored on the predetermined cost-push shock. The first order conditions sets the marginal value of the loss function with respect to inflation at the initial date (itself equal to it costate variable, the Lagrange multiplier μ_0) equal to zero. Initial inflation π_0 increases with cost push shock persistence ρ.*

$$\frac{\partial L}{\partial \pi_0} = \mu_0 = 2\left(P_{\pi^2}\pi_0 + P_z z_0\right) = 0 \Rightarrow \pi_0 = -P_{\pi^2}^{-1}P_{\pi z}\cdot z_0$$
$$\pi_0\left(\underset{+}{A_{\pi z}}, \underset{+}{\rho}\right) = \frac{-1}{1-\beta\rho\lambda}\left(A_{\pi z}\beta\lambda + \frac{Q_{\pi z}}{Q_{\pi^2}}(1-\beta\lambda A)\right)$$
$$\lim_{\rho\to 0}\pi_0 = -A_{\pi z}\beta\lambda - \frac{Q_{\pi z}}{Q_{\pi^2}}(1-\beta\lambda A)$$

Example 5 *As seen in figure 3, with Gali's (2015) calibration ($\rho = 0.8$):*

$$\pi_0 = \frac{\lambda}{1-\lambda\beta\rho} = \frac{0.4291604}{1-0.4291604\cdot 0.99\cdot\rho} = 0.64$$
$$\lim_{\rho\to 0}\pi_0 = \lambda = 0.42274,\ \lim_{\rho\to 1}\pi_0 = \frac{\lambda}{1-\beta\lambda} = 0.73$$

Proposition 6 *Welfare: Even if households and/or Central Bank preferences set a zero weight on the covariance between inflation and cost-push shock ($Q_{\pi z} = 0$) and the variance of cost-push shock ($Q_{zz} = 0$), the variance of the policy instrument depends on this variance. Therefore, the optimal value of welfare depends on the covariance of inflation and of the cost-push shock ($P_{\pi z} \neq 0$) and of the variance of the cost-push shock ($P_{zz} \neq 0$). Both weights ($P_{\pi z}, P_{zz}$) increases with the persistence of the cost-push shock ρ and with the sensitivity of inflation to cost-push shock $A_{\pi z}$. Overall welfare (the opposite of the* *optimal value of the loss function) decreases in a non-linear fashion with the persistence of cost push shock ρ and with the sensitivity of inflation to cost-push shock $A_{\pi z}$ with no response of the policy instrument to the cost-push shock. Welfare can be computed for a given initial predetermined inflation π_0 as follows. For non-predetermined inflation, we take into account the optimal initial anchor of inflation for the welfare of Ramsey optimal policy, using $\pi_0 = -P_{\pi^2}^{-1} P_{\pi z} z_0$:*

$$W\left(\underset{-}{A_{\pi z}}, \underset{-}{\rho}\right) = -\left(\begin{array}{cc} -\frac{P_{\pi z}}{P_{\pi^2}} z_0 & z_0 \end{array}\right) \left(\begin{array}{cc} P_{\pi^2} & P_{\pi z} \\ P_{\pi z} & P_{z^2} \end{array}\right) \left(\begin{array}{c} -\frac{P_{\pi z}}{P_{\pi^2}} z_0 \\ z_0 \end{array}\right) = -\left(P_{z^2} - \frac{P_{\pi z}^2}{P_{\pi^2}}\right) z_0^2 > 0$$

Example 6 *For welfare dependence on the persistence of the cost-push shock, with Gali's (2015) calibration with $\rho = 0.8$ (figure 4):*

$$\frac{W}{z_0^2} = \frac{-\lambda}{(1-\beta\lambda\rho)^2 (1-\beta\rho^2)} = \frac{-0.4291604}{(1-0.99 \cdot 0.4291604 \cdot \rho)^2} \frac{1}{1-0.99 \cdot \rho^2} = -2.688$$

$$\lim_{\rho \to 0} \frac{W}{z_0^2} = -\lambda = -0.429, \ \lim_{\rho \to 1} \frac{W}{z_0^2} = \frac{-\lambda}{(1-\beta\lambda)^2 (1-\beta)} = -130$$

We conclude with a numerical example where the cost-push includes three AR(1) components with low, medium and high persistence, along with three distinct non-correlated random disturbances η_i with $i = 1, 2, 3$.

$$z_t = \sum_{t=1}^{t=3} z_{i,t}, \text{ with } \rho_1 = 0.4, \ \rho_2 = 0.8, \ \rho_3 = 0.99. \tag{9}$$

These autocorrelation coefficient on a quarterly basis leads to $\rho_1^4 = 0.4^4 < 0.03\%$ of the initial value of the shock after one year, $\rho_2^4 = 0.8^{16} < 0.03\%$ of the initial value of the shock after four years and the last component is persistent for more than four years. These degrees of persistence are roughly in line with Ashley *et al.* (2020) three components. We still use Gali (2015) calibration, set aside for the autocorrelation coefficient. Using Scilab code in the appendix, we find:

Example 7 *For three components of low, medium and high persistence $\rho_1 = 0.4$, $\rho_2 = 0.8$, $\rho_3 = 0.99$ in the cost-push shock, the optimal policy is given by:*

$$x_t = 4.51\pi_t - 5.43z_1 - 6.86z_2 - 7.79z_3$$

3 A novel interpretation of the persistence of Fed funds rate and US inflation

In what follows, we replace the letter of the policy instrument x_t (the output gap used in Gali (2015)) by the usual one for the funds rate i_t.

When estimating Taylor rules where interest rate responds to output gap and inflation using quarterly US data since 1982 to 2007, one always find that including two lags of the Fed funds rate increases the R^2 by at least 25% and eliminates the auto-correlation of the residuals. Ashley et al. (2020) estimates of lags are reported:

$$\text{(I): } i_t = \beta_1 i_{t-1} + \beta_2 i_{t-2} + F_\pi \pi_t + F_x x_t \text{ with}$$
$$\widehat{\beta}_1 = \rho_1 + \rho_2 = 1.43 \text{ and } \widehat{\beta}_2 = \rho_1 \rho_2 = -0.51$$

The two roots related to the lags are $\rho_1 = 0.68$ and $\rho_2 = 0.75$. Perhaps, the lagged dependent variables terms may capture the effects of several omitted persistent variables relevant to policy makers, such as oil or energy prices, housing prices, assets prices, exchange rate fluctuations, world demand of goods....

When estimated with Bayesian methods in multiple equations systems of dynamic stochastic equilibrium models, researchers often find one lag of the interest rate in the Taylor rule and an auto-regressive monetary policy shock, which sometimes close to a unit root.

$$\text{(II): } \begin{cases} i_t = \rho_2 i_{t-1} + F_\pi \pi_t + F_x x_t + F_{z_1} z_{1,t} \\ z_{1,t} = \rho_1 z_{1,t-1} + \eta_{1,t} \end{cases}$$

A third specification proposed in this paper is that the policy instrument responds to *two unobserved persistent shocks which have an effect on the policy targets*:

$$\text{(III): } \begin{cases} i_t = F_\pi \pi_t + F_x x_t + F_{z_1} z_{1,t} + F_{z_2} z_{2,t} \\ z_{1,t} = \rho_1 z_{1,t-1} + \eta_{1,t} \\ z_{2,t} = \rho_2 z_{2,t-1} + \eta_{2,t} \end{cases}$$

These three specifications are observationally equivalent while being "*ontologically*" different (Chatelain and Ralf (2018)). Because of observational equivalence, it is impossible to decide which specification is to be preferred, unless one finds an additional criterion for *interpreting* each of these equations as a plausible outcome. In what follows, our criterion is to check whether the proposed "*ontological interpretations*" of the specifications (I), (II) or (III) fits with a linear quadratic optimal and rational behavior of the policy maker related to an optimal program with equations (1) and (2).

For specification (I), it is sometimes erroneously claimed that these two lags of the policy instruments are a *necessary* outcome of linear quadratic optimization including a policy instrument (interest rate) smoothing term in the loss function $R\beta^t i_t^2$ with $R > 0$. The interest rate is computed in deviation from its long run equilibrium value. But the benchmark solution of linear quadratic optimization is a proportional negative-feedback rule such as equation (3), which does not include a lag of the policy instrument. For a single policy target, if the cost push shock is not persistent ($\rho = 0$), the optimal response is $i_t = F_\pi(R)\pi_t$. As known since the early sixties in control theory (Stengel (1994)), a larger weight R on the policy instrument in the quadratic loss function implies a lower response $F_\pi(R)$. This is sufficient to force a lower discounted volatility of the optimal path of the policy instrument in the loss function, weight by R: $R\sum_{t=0}^{T}\beta^t i_r^2$. So it is not necessary to include a lag of the policy instrument for linear-quadratic optimization based on equations (1) and (2). To be exhaustive, if the loss function involves the square of the first and second differences of funds rate $R\beta^t\left((i_t - i_{t-1})^2 + (i_t - i_{t-2})^2\right)$, it may be the case that two lags of the funds are optimal in the policy rule.

For specification (II), the usual interpretation is that the AR(1) shock is a persistent monetary policy shock. The central banker is artificially creating a large persistence of its policy instrument, *independently* of the persistence of the shocks on the policy target.

This behavior artificially increases the discounted volatility of the policy instrument. It is a waste of resources which is sub-optimal as soon as the parameter R is strictly positive. From a rational central banker, this interpretation is only valid if he has a near-zero weight ($R \approx 0$) on the volatility of the policy instrument.

For specification (III), first one can check that taking into account the transmission mechanism, a policy rule responding to one unobserved autoregressive process can be written as a policy rule which does not respond to this process but includes a lag of the policy instrument (Chatelain and Ralf (2019)). This is actually the policy rule put forward by Gali (2015, chapter 5), which is shown to be observationally equivalent to the proportional rule (2) in Chatelain and Ralf (2019). If the policy instrument responds to two distinct unobserved autoregressive process, it is observationally equivalent to a policy rule where the policy instrument does not respond to these two unobserved autoregressive process but where it depends on two lags of the policy instrument (Chatelain and Ralf (2018)).

As demonstrated in this paper, an optimal policy rule always responds to autoregressive shocks in order to reduce the correlation of the policy target with the autoregressive shock. This point is not understood in the current DSGE literature, where an auto-regressive shock for the interest rate rule is interpreted as an autonomous autoregressive "monetary policy shock". This auto-regressive shock is never understood as a rational response to exogenous auto-regressive shocks included in the equations of policy targets in the transmission mechanism. It is an additional auto-regressive shock.

If these auto-regressive shocks are omitted variables, then their persistence fall into the lags of the dependent variable which is the policy instrument. For specification (III), the persistence of the policy instrument is explained by the joint linear-quadratic optimization with interest rate smoothing $R\beta^t i_t^2$ *AND* the substitution of the response of the policy instrument to unobserved persistent shocks on policy targets by lags of the policy instrument.

Hence, the linear quadratic optimal behavior is a consistent *interpretation* of the specification (III), which is observationally equivalent to specifications (I) and (II).

As seen in the impulse response example, when a component of the cost-push shock is highly persistent, the policy instrument can remain highly persistent (one does not reject a unit root hypothesis) while one of the policy target is stationary and converging to its steady state (one does reject a unit root hypothesis). Often, one does not reject a unit root hypothesis with estimations (I) for the funds rate during the period 1982 to 2007. Fuhrer (2010) documented that various quarterly measure of US inflation turned to be stationary since 1982 up to 2007.

The linear quadratic optimal behavior facing highly persistent exogenous shocks is a consistent *interpretation* of having a policy instrument not stationary while the policy target is stationary. Other interpretations are possible.

4 Conclusion

Optimal policy facing persistent exogenous cost-push shock, for example, imported inflation or deflation due to oil or energy shocks implies a dependence to the persistence of the cost-push shock firstly of the policy instrument in the policy rule, secondly of the policy target persistence and of its impulse responses function through a change of its sensitivity to the cost push shock and of the initial jump of inflation and thirdly of welfare.

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5 Appendix

Stable subspace of the Hamiltonian system

Following Chatelain and Ralf (2020), we form the Lagrangian by attaching a sequence of Lagrange multipliers $\beta^{t+1}\mu_{t+1}$ and $\beta^{t+1}\nu_{t+1}$ to the sequence of constraints of the policy transmission mechanism:

$$L = -\sum_{t=0}^{t=+\infty} \beta^t \begin{bmatrix} \frac{1}{2}Q_{\pi^2}\pi_t^2 + Q_{\pi z}\pi_t z_t + \frac{1}{2}Q_{z^2}z_t^2 + \frac{1}{2}Rx_t^2 \\ +\beta\mu_{t+1}\left(A\pi_t + Bx_t + A_{\pi z}z_t - \pi_{t+1}\right) \\ +\beta\nu_{t+1}\left(\rho z_t - z_{t+1}\right) \end{bmatrix}$$

The first order necessary conditions are:

$$\begin{aligned} \frac{\partial L}{\partial x_t} &= Rx_t + \beta B\mu_{t+1} = 0 \Rightarrow x_t = \frac{-\beta B}{R}\mu_{t+1} \text{ or } \mu_{t+1} = -\frac{R}{\beta B}x_t \\ \frac{\partial L}{\partial \pi_t} &= Q_{\pi^2}\pi_t + Q_{\pi z}z_t + \beta A\mu_{t+1} - \mu_t = 0 \\ \frac{\partial L}{\partial z_t} &= Q_{\pi z}\pi_{tt} + Q_{z^2}z_t + \beta A_{\pi z}\mu_{t+1} + \beta\rho v_{t+1} - v_t = 0 \end{aligned}$$

The Hamiltonian system is:

$$\begin{pmatrix} 1 & 0 & \frac{\beta B^2}{R} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta A & 0 \\ 0 & 0 & \beta A_{\pi z} & \beta\rho \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ z_{t+1} \\ \gamma_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} A & A_{\pi z} & 0 & 0 \\ 0 & \rho & 0 & 0 \\ -Q_{\pi^2} & -Q_{\pi z} & 1 & 0 \\ -Q_{z\pi} & -Q_{z^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \\ \gamma_t \\ \mu_t \end{pmatrix}$$

We seek the elements of the value function (welfare) matrix: P_{π^2}, $P_{\pi z}$ and P_{z^2}, which are the unknown parameters of eigenvectors of the stable subspace of the Hamiltonian system:

$$\begin{pmatrix} \pi_t \\ z_t \\ \gamma_t \\ \mu_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ P_{\pi^2} & P_{\pi z} & 0 & 0 \\ P_{z\pi} & P_{z^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \\ \gamma_t \\ \mu_t \end{pmatrix}$$

It follows:

$$\begin{pmatrix} \beta\frac{B^2}{R}P_{\pi^2} + 1 & \beta\frac{B^2}{R}P_{\pi z} \\ 0 & 1 \\ \beta A P_{\pi^2} & \beta A P_{\pi z} \\ \beta\rho P_{\pi z} + \beta A_{\pi z}P_{\pi^2} & \beta\rho P_{z^2} + \beta A_{\pi z}P_{\pi z} \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} A & A_{\pi z} \\ 0 & \rho \\ P_{\pi^2} - Q_{\pi^2} & P_{\pi z} - Q_{\pi z} \\ P_{\pi z} - Q_{\pi z} & P_{z^2} - Q_{z^2} \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix}$$

We eliminate z_{t+1} by ρz_t. *Solving the model amounts to use the following three key equations for finding the three unknown parameters* P_{π^2}, $P_{\pi z}$ and P_{z^2} of the welfare matrix:

$$\begin{pmatrix} 1 + \beta\frac{B^2}{R}P_{\pi^2} \\ \beta A P_{\pi^2} \\ \beta\rho P_{\pi z} + \beta A_{\pi z}P_{\pi^2} \end{pmatrix} (\pi_{t+1}) = \begin{pmatrix} A & A_{\pi z} - \beta\rho\frac{B^2}{R}P_{\pi z} \\ P_{\pi^2} - Q_{\pi^2} & (1 - \beta\rho A)\, P_{\pi z} - Q_{\pi z} \\ P_{\pi z} - Q_{\pi z} & (1 - \beta\rho^2)\, P_{z^2} - Q_{z^2} - \beta\rho A_{\pi z}P_{\pi z} \end{pmatrix} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix} \tag{10}$$

The first terms of the three equations implies three formulas for the intrinsic persistence of the policy target, $\lambda = A + BF_{\pi}$:

$$\lambda = A + BF_{\pi} = \frac{A}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{P_{\pi^2} - Q_{\pi^2}}{A\beta P_{\pi^2}} = \frac{P_{\pi z} - Q_{\pi z}}{\beta \rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}.$$

The second terms of the three equations implies three formulas for the sensitivity of the policy target to the persistent shock, $\delta = A_{\pi z} + BF_{\pi z}$, with $F_{\pi z} = \frac{\delta - A_{\pi z}}{B}$.

$$\delta = \frac{A_{\pi z} - \rho \frac{\beta B^2}{R} P_{\pi z}}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{(1 - \beta \rho A) P_{\pi z} - Q_{\pi z}}{\beta A P_{\pi^2}} = \frac{(1 - \beta \rho^2) P_{z^2} - Q_{z^2} - \beta \rho A_{\pi z} P_{\pi z}}{\beta \rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}.$$

Proof of proposition 1. Compute the optimal root $\lambda = A + BF_{\pi}$, a first element of the welfare matrix P_{π^2} (proposition 5a) and the policy rule parameter F_{π}:

The first term of each of the three equation has the same value:

$$\lambda = \frac{A}{1 + \frac{\beta B^2}{R} P_{\pi^2}} = \frac{P_{\pi^2} - Q_{\pi^2}}{A\beta P_{\pi^2}} = \frac{P_{\pi z} - Q_{\pi z}}{\beta \rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}$$

Using the first equality:

$$1 + \frac{\beta B^2}{R} P_{\pi^2} = \frac{A}{\lambda} \Leftrightarrow P_{\pi^2} = \frac{R}{\beta B^2} \left(\frac{A - \lambda}{\lambda} \right) > 0$$

Using the second equality:

$$P_{\pi^2} - Q_{\pi^2} = \lambda A \beta P_{\pi^2} \Leftrightarrow P_{\pi^2} = \frac{Q_{\pi^2}}{1 - \beta A \lambda}$$

Using the first and second equality leads to a characteristic polynomial for solving λ:

$$\lambda \left(1 + \frac{\beta B^2}{R} P_{\pi^2} \right) - A = 0$$
$$\lambda \left(1 + \frac{B^2}{R} \beta \frac{Q_{\pi^2}}{1 - A\beta\lambda} \right) - A = 0$$
$$\lambda^2 - \left(A + \frac{1}{A\beta} + \frac{B^2 Q_{\pi^2}}{AR} \right) \lambda + \frac{1}{\beta} = 0$$

Optimal persistence is the stable root of this characteristic polynomial:

$$0 < \lambda = \frac{1}{2} \left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R} \frac{B^2}{A} - \sqrt{\left(A + \frac{1}{\beta A} + \frac{Q_{\pi^2}}{R} \frac{B^2}{A} \right)^2 - \frac{4}{\beta}} \right)$$
$$0 < \lambda \leq \min \left(A, \frac{1}{\beta A} \right) \text{ for } Q_{\pi^2} = 0$$

The policy rule parameter is a function of the optimal persistence λ:

$$F_{\pi} = \frac{\lambda - A}{B} \text{ and } BF_{\pi} < 0 \text{ so that } \lambda = A + BF_{\pi} \leq A.$$

Proof of proposition 5b: Second element of the welfare matrix $P_{\pi z}$
We use the third equality for the first term:

$$\lambda = \frac{A}{1+\frac{\beta B^2}{R}P_{\pi^2}} = \frac{P_{\pi^2}-Q_{\pi^2}}{A\beta P_{\pi^2}} = \frac{P_{\pi z}-Q_{\pi z}}{\beta\rho P_{\pi z}+\beta A_{\pi z}P_{\pi^2}}$$

$P_{\pi z}$ is an increasing function of the two characteristics of the forcing variable $A_{\pi z}$ and ρ:

$$\begin{aligned} P_{\pi z}-Q_{\pi z} &= \lambda\beta\rho P_{\pi z}+\lambda\beta A_{\pi z}P_{\pi^2} \Rightarrow \\ P_{\pi z} &= \frac{\beta\lambda A_{\pi z}P_{\pi^2}+Q_{\pi z}}{1-\beta\rho\lambda} \end{aligned}$$

With:

$$P_{\pi^2} = \frac{R}{\beta B^2}\left(\frac{A-\lambda}{\lambda}\right) = \frac{Q_{\pi^2}}{1-A\beta\lambda}$$

$P_{\pi z}$ can be written as a function of λ:

$$\begin{aligned} P_{\pi z}(A_{\pi z},\rho) &= \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}\frac{R}{\beta B^2}\beta\lambda\left(\frac{A-\lambda}{\lambda}\right)+Q_{\pi z}\right) \\ P_{\pi z}(A_{\pi z},\rho) &= \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}\frac{R}{B^2}(A-\lambda)+Q_{\pi z}\right) \quad \text{or} \\ P_{\pi z}(A_{\pi z},\rho) &= \frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}Q_{\pi^2}\frac{\beta\lambda}{1-A\beta\lambda}+Q_{\pi z}\right) \end{aligned}$$

Proof of proposition 2: Compute the sensitivity $\delta = A_{\pi z}+BF_z$:
One has:

$$\delta = \frac{A_{\pi z}-\beta\rho\frac{B^2}{R}P_{\pi z}}{\frac{A}{\lambda}} = \frac{(1-A\beta\rho)P_{\pi z}-Q_{\pi z}}{\beta AP_{\pi^2}} = \frac{(1-\beta\rho^2)P_{z^2}-Q_{z^2}-\beta\rho A_{\pi z}P_{\pi z}}{\beta\rho P_{\pi z}+\beta A_{\pi z}P_{\pi^2}}.$$

In the first equality, substitute $P_{\pi z}$ and the denominator by A/λ leads to:

$$\begin{aligned} \delta &= \frac{\lambda}{A}\left(A_{\pi z}-\frac{\beta\rho}{1-\beta\lambda\rho}\frac{B^2}{R}\left(A_{\pi z}\frac{R}{B^2}(A-\lambda)+Q_{\pi z}\right)\right) \\ \delta &= \frac{\lambda}{A}\left(A_{\pi z}\left(1-\frac{\beta\rho}{1-\beta\lambda\rho}(A-\lambda)\right)-Q_{\pi z}\frac{B^2}{R}\frac{\beta\rho}{1-\beta\lambda\rho}\right) \\ \delta &= \frac{\lambda}{A}\left(A_{\pi z}\frac{1-A\beta\rho}{1-\lambda\beta\rho}-Q_{\pi z}\frac{\beta B^2}{R}\frac{\rho}{1-\beta\lambda\rho}\right) \end{aligned}$$

Proof of example 2: For Gali's example: $Q_{\pi z}=0$ and $A=-A_{\pi z}=1/\beta$

$$\delta = \frac{1}{\beta}\left(1-\frac{1-\beta\lambda}{1-\beta\lambda\rho}\right) = (1-\rho)\frac{-\lambda}{1-\beta\lambda\rho}$$

Proof of proposition 3: The impulse response function of inflation

The following result can be found using at least three methods:

$$\pi_t = \lambda^t \left(\pi_0 - \frac{(A_{\pi z} + BF_z)\, z_0}{\rho - \lambda} \right) + \rho^t \frac{(A_{\pi z} + BF_z)\, z_0}{\rho - \lambda}$$

(a) Solve the sum of two geometric sequences or geometric progressions using the homogeneous solution related to the geometric sequence with common ratio λ and a particular solution proportional to the forcing variable following a geometric sequence with common ratio ρ;

(b) compute the power of the matrix $\begin{pmatrix} \lambda & A_{\pi z} + BF_z \\ 0 & \rho \end{pmatrix}$ using its Jordan decomposition;

(c) prove it by mathematical induction.

Proof of proposition 4: Policy rule parameter F_z:

$$F_z = \frac{\delta - A_{\pi z}}{B} = \frac{A_{\pi z}}{B} \left(\frac{\lambda - A\lambda\beta\rho}{A - A\lambda\beta\rho} - 1 \right) - Q_{\pi z} \frac{B}{R} \frac{\beta\lambda\rho}{A - A\beta\lambda\rho}$$

$$F_z = -\frac{A_{\pi z}}{AB} \left(\frac{A - \lambda}{1 - \lambda\beta\rho} \right) - Q_{\pi z} \frac{B}{R} \frac{\beta\lambda\rho}{A - A\beta\lambda\rho}$$

$$F_z = \frac{A_{\pi z}}{A} \frac{1}{1 - \beta\lambda\rho} \frac{\lambda - A}{B} - \frac{Q_{\pi z}}{A} \frac{B}{R} \frac{\beta\lambda\rho}{1 - \beta\lambda\rho}$$

Proof of example 4: For Gali's example: $Q_{\pi z} = 0$ and $A = -A_{\pi z} = 1/\beta$

$$F_z = \frac{1}{\kappa} \left(-\frac{1 - \beta\lambda}{1 - \beta\lambda\rho} \right)$$

Proof of proposition 5c: Optimal jump of inflation π_0:

The first order conditions sets the marginal value of the loss function with respect to inflation at the initial date (itself equal to it costate variable, the Lagrange multiplier μ_0) equal to zero.

$$\frac{\partial L}{\partial \pi_0} = \mu_0 = 2\left(P_{\pi^2}\pi_0 + P_z z_0\right) = 0 \Rightarrow \pi_0 \left(\underset{+}{A_{\pi z}}, \underset{+}{\rho} \right) = -\frac{P_{\pi z}}{P_{\pi^2}} \cdot z_0$$

One has:

$$-\frac{P_{\pi z}}{P_{\pi^2}} = \frac{-1}{P_{\pi^2}} \frac{Q_{\pi z} + A_{\pi z}\beta\lambda P_{\pi^2}}{1 - \beta\rho\lambda} \text{ and } P_{\pi^2} = \frac{Q_{\pi^2}}{1 - \beta\lambda A}$$

$$-\frac{P_{\pi z}}{P_{\pi^2}} = \frac{-1}{1 - \beta\rho\lambda} \left(A_{\pi z}\beta\lambda + \frac{Q_{\pi z}}{Q_{\pi^2}} (1 - \beta\lambda A) \right)$$

Proof of proposition 6: Welfare parameter P_{z^2}

The third equation related to the sensitivity of the policy target δ to the persistent shock determines P_{z^2}:

$$\delta = \frac{\lambda}{A} \left(A_{\pi z} - \rho \frac{\beta B^2}{R} P_{\pi z} \right) = \frac{(1 - A\beta\rho)\, P_{\pi z} - Q_{\pi z}}{\beta A P_{\pi^2}} = \frac{(1 - \beta\rho^2)\, P_{z^2} - Q_{z^2} - \beta\rho A_{\pi z} P_{\pi z}}{\beta\rho P_{\pi z} + \beta A_{\pi z} P_{\pi^2}}$$

The third equality leads to:

$$\left(1-\beta\rho^2\right)P_{z^2}=Q_{z^2}+\left(A_{\pi z}+\delta\right)\beta\rho P_{\pi z}+\delta\beta A_{\pi z}P_{\pi^2}$$

with:

$$P_{\pi z}=\frac{1}{1-\beta\lambda\rho}\left(A_{\pi z}\beta\lambda\frac{Q_{\pi^2}}{1-\lambda\beta A}+Q_{\pi z}\right)\text{ and }P_{\pi^2}=\frac{Q_{\pi^2}}{1-\lambda\beta A}$$

$$A_{\pi z}+\delta=A_{\pi z}+\frac{\left(1-A\beta\rho\right)P_{\pi z}-Q_{\pi z}}{A\beta P_{\pi^2}}$$

so that:

$$\begin{pmatrix} P_{\pi^2} & P_{\pi z}\\ P_{\pi z} & P_{z^2}\end{pmatrix}=\begin{pmatrix}\frac{Q_{\pi 2}}{1-\beta\lambda A} & \frac{Q_{\pi z}}{1-\beta\lambda\rho}\\ \frac{Q_{\pi z}}{1-\beta\lambda\rho} & \frac{Q_{z 2}}{1-\beta\rho^2}\end{pmatrix}$$
$$+\begin{pmatrix} 0 & \frac{1}{1-\beta\lambda\rho}A_{\pi z}\beta\lambda P_{\pi^2}\\ \frac{1}{1-\beta\lambda\rho}A_{\pi z}\beta\lambda P_{\pi^2} & \frac{1}{1-\beta\rho^2}\left(\left(A_{\pi z}+\delta\right)\beta\rho P_{\pi z}+\delta\beta A_{\pi z}P_{\pi^2}\right)\end{pmatrix}$$

Proof of welfare, example 6 (Gali): Computation of P_{z^2}:

$$\left(1-\beta\rho^2\right)P_{z^2}=Q_{z^2}+\delta\left(\beta\rho P_{\pi z}+\beta A_{\pi z}P_{\pi^2}\right)+\beta A_{\pi z}\rho P_{\pi z}$$

with:

$$P_{\pi z}=\frac{-\lambda}{1-\beta\rho\lambda}\frac{1}{1-\lambda}\text{ and }P_{\pi^2}=\frac{1}{1-\lambda}$$
$$\delta=(1-\rho)\left(\frac{-\lambda}{1-\beta\lambda\rho}\right),\ Q_{z^2}=0,\ \beta A_{\pi z}=-1$$

One has:

$$\left(1-\beta\rho^2\right)P_{z^2}=\frac{-\lambda}{1-\beta\lambda\rho}\left((1-\rho)\left(\beta\rho\left(\frac{-\lambda}{1-\beta\rho\lambda}\frac{1}{1-\lambda}\right)-\frac{1}{1-\lambda}\right)-\rho\frac{1}{1-\lambda}\right)$$
$$\left(1-\beta\rho^2\right)P_{z^2}=\frac{\lambda}{1-\lambda}\frac{-1}{1-\beta\lambda\rho}\left(-\frac{\beta\lambda\rho^2-1}{\beta\lambda\rho-1}\right)$$
$$P_{z^2}=\frac{1}{1-\lambda}\frac{\lambda}{\left(1-\lambda\beta\rho\right)^2}\frac{1-\lambda\beta\rho^2}{1-\beta\rho^2}$$

so that:

$$\mathbf{P}=\begin{pmatrix}\frac{1}{1-\lambda} & -\frac{1}{1-\lambda}\frac{\lambda}{1-\beta\rho\lambda}\\ -\frac{1}{1-\lambda}\frac{\lambda}{1-\beta\rho\lambda} & \frac{1}{1-\lambda}\frac{\lambda}{(1-\beta\rho\lambda)^2}\frac{1-\beta\lambda\rho^2}{1-\beta\rho^2}\end{pmatrix}=\begin{pmatrix}1.7518055 & -1.1389181\\ -1.1389181 & 3.4285107\end{pmatrix}$$

For welfare dependence on the persistence of the cost-push shock, with Gali's (2015) calibration:

$$P_{z^2} = \frac{1}{1-\lambda}\frac{\lambda}{(1-\beta\rho\lambda)^2}\frac{1-\lambda\beta\rho^2}{1-\beta\rho^2} = 3.4285$$

$$-\frac{P_{\pi z}^2}{P_{\pi^2}} = -\frac{1}{1-\lambda}\frac{\lambda^2}{(1-\beta\lambda\rho)^2}$$

Welfare has this form:

$$\frac{W}{z_0^2} = -P_{z^2} + \frac{P_{\pi z}^2}{P_{\pi^2}} = \frac{1}{1-\lambda}\frac{\lambda}{(1-\beta\rho\lambda)^2}\left(\lambda - \frac{1-\beta\lambda\rho^2}{1-\beta\rho^2}\right)$$

$$\frac{W}{z_0^2} = \frac{\lambda}{1-\lambda}\frac{-1}{(1-\beta\rho\lambda)^2}\frac{1-\lambda}{1-\beta\rho^2}$$

$$\frac{W}{z_0^2} = \frac{-\lambda}{(1-\lambda\beta\rho)^2}\frac{1}{1-\beta\rho^2} = \frac{-0.4291604}{(1-0.99\cdot\rho\cdot 0.4291604)^2}\frac{1}{1-0.99\cdot\rho^2}$$

$$\frac{W}{z_0^2} = \frac{-0.4291604}{(1-0.99\cdot 0.8\cdot 0.4291604)^2}\frac{1}{1-0.99\cdot 0.8^2} = 2.688$$

QED.

SCILAB Code for numerical solutions:

```
beta1=0.99; eps=6; kappa=0.1275; rho=0.4;
Qpi=1; Qz=0 ; Qzpi=0; R=kappa/eps;
A1=[1/beta1 -1/beta1 ; 0 rho] ;
A=sqrt(beta1)*A1;
B1=[-kappa/beta1 ; 0];
B=sqrt(beta1)*B1;
Q=[Qpi Qzpi ;Qzpi Qz ];
Big=sysdiag(Q,R);
[w,wp]=fullrf(Big);
C1=wp(:,1:2);
D12=wp(:,3:$);
M=syslin('d',A,B,C1,D12);
[Fy,Py]=lqr(M)
A1+B1*Fy
-inv(Py(1,1))*Py(1,2)
Py(2,2)-Py(1,2)*inv(Py(1,1))*Py(1,2)
```

Figure 2a: Impulse response function during following a shock of 1 unit, with persistence ρ=0.99

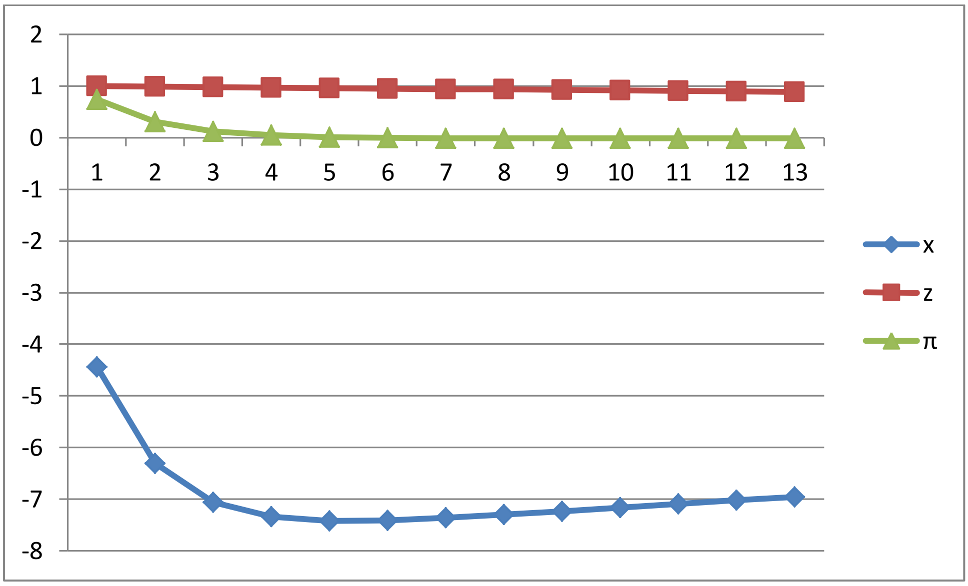

Figure 2b: Impulse response function during following a shock of 1 unit, with persistence ρ=0.8