The term structure of euro area sovereign bond yields*

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31 May 2012

Preliminary and incomplete

Abstract

We model the dynamics of bond yields across a number of euro area sovereigns as functions of a default-free part and a component that is due to credit risk. The first part reflects investors' perceptions about expected ECB monetary policy (and associated term premia), while the second is largely associated with the perceived risk of sovereign default and other credit events in credit-risky countries. We let this latter credit spread component be a function of the perceived fiscal situation of each country, which in turn is assumed to be filtered out from deficit/GDP forecast data, and of country-specific GDP growth. We allow the fiscal variable to affect sovereign yields and spreads in a non-linear way using an affine-quadratic model. We also include a latent common credit factor which can affect spreads across all credit-risky sovereigns, in order to capture common systematic or contagion effects. We find that our model can fit observed data well, including the extreme yield levels seen in some countries in recent months. We decompose the credit spreads of sovereigns bonds into a credit risk premium and a default risk component and examine their relative importance.

JEL classification numbers: F34, G12, G15

Keywords: Sovereign bond yields, affine quadratic term structure, fiscal policy, credit risk, reduced form credit models.

*We would like to thank ... for helpful comments and suggestions. The opinions expressed are personal and should not be attributed to the Bank for International Settlements or the European Central Bank.

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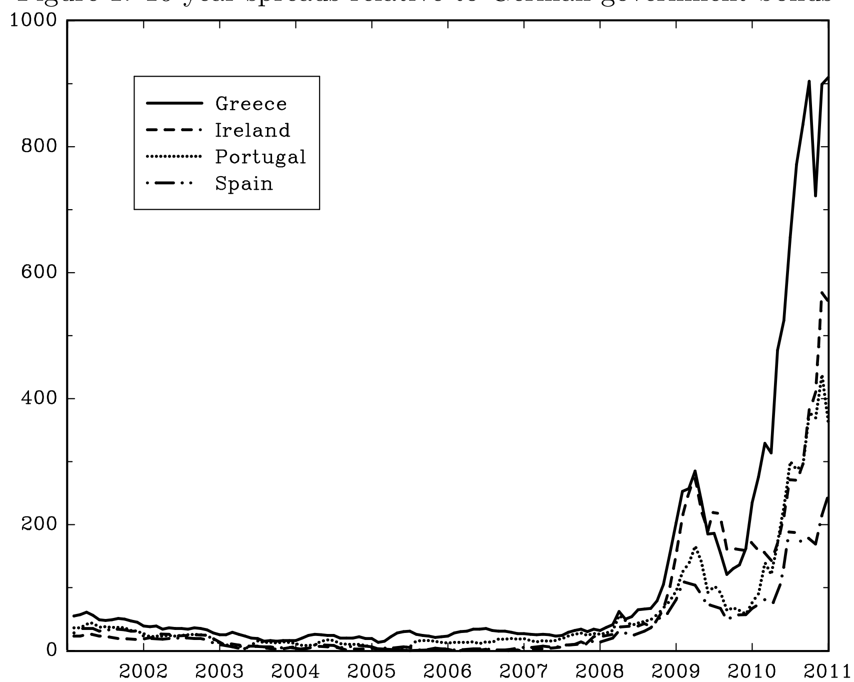
1 Introduction

From late 2009 onwards, yields on bonds issued by peripheral euro area sovereigns rose sharply above comparable yields on German government bonds. Credit spreads for Greece, Ireland, Portugal and Spain, which had averaged only a few tens of basis points for most of the period since the introduction of the euro, surged to several hundred basis points. The underlying reason was rapidly eroding confidence in the ability of these governments to be able to meet commitments to their lenders as budget deficits and debt-to-GDP ratios jumped.

Figure 1: 10-year spreads relative to German government bonds



In annual basis points. Source: ECB.

Although it seems clear that there were fundamental fiscal concerns for a number of peripheral euro area countries, the high degree of co-movement in the recent spread widening, visible in Figure 1, suggests that perhaps not only country-specific fiscal factors were important in pushing spreads up. In this paper, we investigate the underlying factors driving the recent spread widening by modeling the dynamics of the term structure of sovereign credit spreads in a way that allows both country-specific and common factors to systematically affect credit spreads. To capture country-specific effects, we include a deficit/GDP variable and a GDP growth variable that we filter out from available forecasts of these variables for each of the countries under consideration. To allow for the possibility that there may be spillover effects or global fundamental drivers of spreads, we include a latent common factor that can affect spreads for all credit-risky countries.

We explicitly model the German term structure of interest rates, which we take to be free of default (or other sovereign credit) risk, or at least perceived by investors to be free of such risk. German yields are assumed to reflect average expected interest rates set by the European Central Bank (ECB), plus term premia to compensate for the associated uncertainty. Specifically, we model the perceived interest rate setting decisions of the ECB as a Taylor rule that depends on euro area inflation and output gap, as well as on monetary policy shocks. This setup makes it possible for us to examine the effects of policy shocks on sovereign spreads, through their impact on the output gap and hence on fiscal sustainability.

Our modeling approach is related to a number of papers that price credit risky securities using a reduced-form on-arbitrage approach, e.g. Duffie, Pedersen and Singleton (2003). We differ from their setting, as well as that in most of the litterature, in a number of ways. First, we include observable macro variables, including fiscal information, and we consider several sovereigns jointly. Amato and Luisi (2006) also include macro factors within a reduced-form model of credit spreads, although they consider US corporates rather than sovereign entities. Dai and Philippon (2006) examine the US term structure of interest rates using a model that includes fiscal information in addition to other macro variables. While they find significant effects of fiscal variables on bond yields, these effects are, in their modelling framework, not at all related to credit risk, given that they explicitly assume that US bonds are risk-free. Here, instead, we view changes in spreads as reflecting variations in the perceived sovereign credit risk, and in the compensation required by investors to bear such risk. Our work is also related to Pan and Singleton (2008) and Longstaff, Pan, Pedersen and Singleton (2010), who examine sovereign credit risk using data on sovereign credit default swaps (CDS). They estimate a single-factor model for each of the sovereigns they consider and then investigate how credit risk covaries across countries. By contrast, we jointly estimate the credit spreads for four euro area peripheral countries, where we explicitly allow for a common factor. Our paper is closest to that of Borgy et al. (2011), who also study euro area credit spreads using observable fiscal information. Compared to their paper, we differ by including a euro-wide common credit factor that potentially can capture contagion effects on spreads, and by extending the bond pricing framework from a linear to a non-linear setting. Specifically, we allow the default intensity of a credit-risky country to depend on our fiscal forecast variable in a linear-quadratic way. As a result, we end up with an affine-quadratic pricing framework, which we find to work well in capturing some of the extreme sovereign credit spread widening witnessed in recent months.

[more references to be added]

This paper is organised as follows. Section 2 presents our modelling approach and discusses the estimation method. Section 3 describes the data. Section 4 presents the results for the benchmark default-free term structure and for the credit-risky term structures. Section 5 concludes.

2 Model and estimation

Our empirical specification is based on the class of reduced-form credit pricing models in which assumptions are made about the process for default intensity, as in Lando (1998) and Duffie and Singleton (1999). In this framework, default is assumed to be doubly stochastic, meaning that default arrives randomly according to a Poisson process with time-varying intensity and, in addition, this intensity process varies randomly over time. The advantage of this approach is that it gives rise to tractable pricing formulas. Specifically, in continuous time, for a given risk-neutral default intensity process $\psi^{\mathbb{Q}}$ and a given risk-free interest rate process r, the price at t of a zero-coupon defaultable bond with maturity T and zero recovery is (see e.g. Duffie and Singleton, 2003):

$$d(t,T) = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r(u) + \psi^{\mathbb{Q}}(u)\right) du\right)\right].$$

In case of non-zero (possibly stochastic) recovery Z, Lando (1998) shows that the price is

$$\begin{aligned} d(t,T) &= E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T \left(r(u) + \psi^{\mathbb{Q}}(u)\right) du\right)\right] + \\ & E_t^{\mathbb{Q}}\left[\int_t^T Z(s)\,\psi^{\mathbb{Q}}(s)\exp\left(-\int_t^s \left(r(u) + \psi^{\mathbb{Q}}(u)\right) du\right) ds\right], \end{aligned}$$

where $Z = 1 - L^{\mathbb{Q}}$ and $L^{\mathbb{Q}}$ is the risk-neutral loss rate in the case of default.

In some cases, notably under the assumption of fractional recovery of the market value (RMV) of the bond, it is possible to obtain closed-form solutions for defaultable bonds (Duffie and Singleton, 1999). We assume RMV and proceed by setting up our empirical specification in discrete time.

2.1 Model: Basic setup

We specify our model as a Gaussian macro-finance term structure model, in the spirit of Ang and Piazzesi (2003).[1] The state vector, denoted X_t, which will contain both observable and unobservable factors, follows an AR(1):

$$X_t = \Phi X_{t-1} + \Sigma \varepsilon_t, \tag{1}$$

where it is assumed that the state variables are expressed in deviations from their mean. In addition, we assume that the risk-free short-term interest rate r_t is affine in (some of) the state variables,

$$r_t = \delta_0 + \delta X_t.$$

The pricing kernel $m_{t,t+1}$ is assumed to depend on X_t. Specifically, $m_{t,t+1} = \exp(-r_t)\,\kappa_{t+1}/\kappa_t$, where κ_{t+1} is assumed to follow the log-normal process $\kappa_{t+1} = \kappa_t \exp\left(-\frac{1}{2}\psi_t'\psi_t - \psi_t'\varepsilon_{t+1}\right)$, which results in

$$m_{t,t+1} = \exp\left(-r_t - \frac{1}{2}\psi_t'\psi_t - \psi_t'\varepsilon_{t+1}\right).$$

The vector ψ_t represents the market prices of risk associated with the systematic underlying sources of uncertainty in the economy. We follow Duffee (2002) and assume that the market prices of risk are affine in X_t :

$$\psi_t = \psi_0 + \psi_1 X_t,$$

so that the market's required compensation for bearing risk can vary with the state of the economy.

The default intensities of risky bonds, Λ_t, are assumed to depend on the state variables (or a subset of them) as well. Note that we allow credit-specific state variables to be priced by the market, which means that we cannot in general assume that default intensities (or credit spreads) are independent of the risk-free interesty rate process.

[1] The macro-finance literature has grown tremendously over the past few years. Recent papers include Ang, Dong and Piazzesi (2006), Ang, Piazzesi and Wei (2006), Dewachter and Lyrio, (2006), Hördahl, Tristani and Vestin (2006), and Rudebusch and Wu (2004).

2.2 Risk-free bond prices

Given this setup, the price of an n-period default-free zero coupon bond, P_t^n, can be shown to be given by

$$P_t^n = \exp\left(\bar{A}_n + \bar{B}_n' X_t\right),$$

where $\bar{A}_n$ and $\bar{B}_n'$ are defined recursively as

$$\bar{A}_n = \bar{A}_{n-1} - \bar{B}_{n-1}'\Sigma\psi_0 + \frac{1}{2}\bar{B}_{n-1}'\Sigma\Sigma'\bar{B}_{n-1} - \delta_0, \tag{2}$$

$$\bar{B}_n' = \bar{B}_{n-1}'(\Phi - \Sigma\psi_1) - \delta, \tag{3}$$

with initial conditions

$$\bar{A}_1 = -\delta_0,$$
$$\bar{B}_1' = -\delta.$$

The corresponding continuously compounded yield y_t^n is given by

$$y_t^n = \tilde{A}_n + \tilde{B}_n' X_t, \tag{4}$$

with

$$\tilde{A}_n = -\frac{\bar{A}_n}{n},$$
$$\tilde{B}_n = -\frac{\bar{B}_n}{n}.$$

2.3 Risky bond prices

The price at t of a risky bond maturing at $t+n$ can be written as

$$B_t^{t+n} = E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\mathbf{1}_{\tau>t+1} + Z_{t+1}\mathbf{1}_{\tau<t+1}\right)\right],$$

with boundary condition

$$B_{t+n-1}^{t+n} = E_{t+n-1}\left[m_{t+n-1,t+n}\left(B_{t+n-1}^{t+n}\mathbf{1}_{\tau>t+n} + Z_{t+n}\mathbf{1}_{\tau<t+n}\right)\right],$$

where Z_t is the recovery payment, τ denotes the time of default and $\mathbf{1}_{\tau>t+1}$ is an indicator variable that takes the value one if $\tau > t+1$. In general, the expectation

$E_t\left[\mathbf{1}_{\tau>t+k}\right]$ is the probability of survival until $t+k$:

$$E_t\left[\mathbf{1}_{\tau>t+k}\right] = E_t\left[\exp\left(-\sum_{i=1}^{k}\Lambda_{t+i}\right)\right].$$

Under a RMV assumption, the expected recovery payment is a fraction of the bond price at $t+1$, conditional on no default, ie for an n-period bond

$$E_t\left[Z_{t+1}\right] = E_t\left[\left(1-L_{t+1}\right)B_{t+1}^{t+n}\right],$$

where L_{t+1} is the fractional loss rate.

Assuming that the loss rate is a constant L, we have, under RMV,

$$B_t^{t+n} = E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\mathbf{1}_{\tau>t+1} + (1-L)\,B_{t+1}^{t+n}\mathbf{1}_{\tau<t+1}\right)\right],$$

which can be written

$$\begin{aligned} B_t^{t+n} &= E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\exp\left(-\Lambda_{t+1}\right) + B_{t+1}^{t+n}\left(1-L\right)\left(1-\exp\left(-\Lambda_{t+1}\right)\right)\right)\right] \\ &= E_t\left[m_{t,t+1}\left(\exp\left(-\Lambda_{t+1}\right) + \left(1-L\right)\left(1-\exp\left(-\Lambda_{t+1}\right)\right)\right)B_{t+1}^{t+n}\right] \\ &= E_t\left[m_{t,t+1}\left(1-L\left(1-\exp\left(-\Lambda_{t+1}\right)\right)\right)B_{t+1}^{t+n}\right]. \end{aligned}$$

Assume that we can make the following approximation

$$1-L\left(1-\exp\left(-\Lambda_{t+1}\right)\right) \approx \exp\left(-\Lambda_{t+1}\right).$$

This approximation holds exactly for $L=1$. For L different from 1, we should view Λ as reflecting adjusted default intensities, rather than actual intensities. This analogous to the use of "recovery-adjusted default intensities" in continuous time models with RMV (e.g. Duffie and Singleton, 1999). Given this assumption, we can write

$$B_t^{t+n} = E_t\left[m_{t,t+1}\exp\left(-\Lambda_{t+1}\right)B_{t+1}^{t+n}\right].$$

We will assume that the (adjusted) default intensity of country j is a quadratic function of the states:

$$\Lambda_t^j = \lambda_0^j + \lambda^j X_t + X_t'\Xi^j X_t.$$

The price of an n-period bond is therefore (supressing superscripts j)

$$\begin{aligned}
B_t^{t+n} &= E_t\left[m_{t,t+1}\exp\left(-\Lambda_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1 X_t-\frac{1}{2}X_t'\psi_1'\psi_1 X_t-X_t'\Phi'\Xi\Phi X_t\right) \\
&\quad\times E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)B_{t+1}^{t+n}\right]
\end{aligned}$$

We know, given the affine-quadratic setup, that we can write the price of a bond as

$$B_t^{t+n}=\exp\left(A_n+B_nX_t+X_t'C_nX_t\right).$$

If we plug in

$$\begin{aligned}
B_{t+1}^{t+n} &= \exp\left(A_{n-1}+B_{n-1}X_{t+1}+X_{t+1}'C_{n-1}X_{t+1}\right) \\
&= \exp\left(A_{n-1}+B_{n-1}\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)+\left(X_t'\Phi'+\varepsilon_{t+1}'\Sigma'\right)C_{n-1}\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)\right) \\
&= \exp\left(\begin{array}{c} A_{n-1}+B_{n-1}\Phi X_t+B_{n-1}\Sigma\varepsilon_{t+1} \\ +X_t'\Phi'C_{n-1}\Phi X_t+2X_t'\Phi'C_{n-1}\Sigma\varepsilon_{t+1}+\varepsilon_{t+1}'\Sigma'C_{n-1}\Sigma\varepsilon_{t+1}\end{array}\right)
\end{aligned}$$

into the bond price above we get

$$\begin{aligned}
B_t^{t+n} &= \exp\left(\begin{array}{c} -\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0+A_{n-1}+\left(B_{n-1}\Phi-\delta-\lambda\Phi-\psi_0'\psi_1\right)X_t \\ +X_t'\left(\Phi'C_{n-1}\Phi-\Phi'\Xi\Phi-\frac{1}{2}\psi_1'\psi_1\right)X_t\end{array}\right) \\
&\quad\times E_t\left[\exp\left(\begin{array}{c}\left(B_{n-1}\Sigma+2X_t'\Phi'C_{n-1}\Sigma-\psi_0'-X_t'\psi_1'-\lambda\Sigma-2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right],
\end{aligned}$$

where the expectation can be written as $E_t\left[\exp\left(aw_{t+1}+w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right]$, where

$$\begin{aligned}
w_{t+1} &\equiv \Sigma\varepsilon_{t+1} \\
a &\equiv B_{n-1}+2X_t'\Phi'C_{n-1}-\psi_0'\Sigma^{-1}-X_t'\psi_1'\Sigma^{-1}-\lambda-2X_t'\Phi'\Xi, \\
\bar{C}_{n-1} &\equiv C_{n-1}-\Xi.
\end{aligned}$$

To evaluate the expectation we follow Realdon (2006), who demonstrates that (if γ is of full rank)

$$E_t\left[\exp\left(aw_{t+1}+w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right]=\frac{|\gamma|}{\operatorname{abs}|\Sigma|}\prod_{i=1}^{N}\exp\left(\frac{\left(a\gamma_i\right)^2}{2}\right)$$

where $\gamma \equiv \left((\Sigma\Sigma')^{-1} - 2\bar{C}_{n-1} \right)^{-1/2}$, γ_i denotes the i-th column of γ, $|\gamma|$ denotes the determinant of γ and abs$|\Sigma|$ denotes the absolute value of the determinant of Σ. We therefore get

$$\begin{aligned}
& E_t \left[\exp \left(a w_{t+1} + w'_{t+1} \bar{C}_{n-1} w_{t+1} \right) \right] \\
= \; & \frac{|\gamma|}{\text{abs}\,|\Sigma|} \prod_{i=1}^{N} \exp \left(\frac{\left(\left(B_{n-1} + 2X'_t \Phi' C_{n-1} - \psi'_0 \Sigma^{-1} - X'_t \psi'_1 \Sigma^{-1} - \lambda - 2X'_t \Phi' \Xi \right) \gamma_i \right)^2}{2} \right),
\end{aligned}$$

so that

$$\begin{aligned}
\ln B_t^{t+n} \;=\; & -\delta_0 - \lambda_0 - \frac{1}{2}\psi'_0\psi_0 + A_{n-1} + \left(B_{n-1}\Phi - \delta - \lambda\Phi - \psi'_0\psi_1 \right) X_t \\
& + X'_t \left(\Phi' C_{n-1} \Phi - \Phi' \Xi \Phi - \frac{1}{2}\psi'_1\psi_1 \right) X_t \\
& + \ln \frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2} \sum_{i=1}^{N} \left(\left(B_{n-1} + 2X'_t \Phi' C_{n-1} - \psi'_0 \Sigma^{-1} - X'_t \psi'_1 \Sigma^{-1} - \lambda - 2X'_t \Phi' \Xi \right) \gamma_i \right)^2 .
\end{aligned}$$

Evaluating the squared term we get

$$\begin{aligned}
\ln B_t^{t+n} \;=\; & -\delta_0 - \lambda_0 - \frac{1}{2}\psi'_0\psi_0 + A_{n-1} + \left(B_{n-1}\Phi - \delta - \lambda\Phi - \psi'_0\psi_1 \right) X_t \\
& + X'_t \left(\Phi' C_{n-1} \Phi - \Phi' \Xi \Phi - \frac{1}{2}\psi'_1\psi_1 \right) X_t \\
& + \ln \frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2} \sum_{i=1}^{N} \left(B_{n-1} - \psi'_0 \Sigma^{-1} - \lambda \right) \gamma_i \gamma'_i \left(B'_{n-1} - \Sigma^{-1'} \psi_0 - \lambda' \right) \\
& + \sum_{i=1}^{N} \left(B_{n-1} - \psi'_0 \Sigma^{-1} - \lambda \right) \gamma_i \gamma'_i \left(2C'_{n-1}\Phi - \Sigma^{-1'} \psi_1 - 2\Xi'\Phi \right) X_t \\
& + \frac{1}{2} \sum_{i=1}^{N} X'_t \left(2\Phi' C_{n-1} - \psi'_1 \Sigma^{-1} - 2\Phi'\Xi \right) \gamma_i \gamma'_i \left(2C'_{n-1}\Phi - \Sigma^{-1'} \psi_1 - 2\Xi'\Phi \right) X_t .
\end{aligned}$$

We can therefore identify the recusive factor loadings of the bond price $B_t^{t+n} =$

$\exp\left(A_n + B_n X_t + X_t' C_n X_t\right)$ as

$$
\begin{aligned}
A_n &= A_{n-1} - \delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + \ln\frac{|\gamma|}{\mathrm{abs}\,|\Sigma|} \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(B_{n-1}' - \Sigma^{-1\prime}\psi_0 - \lambda'\right), \\
B_n &= B_{n-1}\Phi - \delta - \lambda\Phi - \psi_0'\psi_1 \\
&\quad + \sum_{i=1}^{N}\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1\prime}\psi_1 - 2\Xi'\Phi\right), \\
C_n &= \Phi' C_{n-1}\Phi - \Phi'\Xi\Phi - \frac{1}{2}\psi_1'\psi_1 \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\left(2\Phi' C_{n-1} - \psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1\prime}\psi_1 - 2\Xi'\Phi\right),
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
A_1 &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + \ln\frac{|\gamma^1|}{\mathrm{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1\prime}\left(-\Sigma^{-1\prime}\psi_0 - \lambda'\right), \\
B_1 &= -\delta - \lambda\Phi - \psi_0'\psi_1 + \sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1\prime}\left(-\Sigma^{-1\prime}\psi_1 - 2\Xi'\Phi\right) \\
C_1 &= -\frac{1}{2}\psi_1'\psi_1 - \Phi'\Xi\Phi + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1\prime}\left(-\Sigma^{-1\prime}\psi_1 - 2\Xi'\Phi\right).
\end{aligned}
$$

The general setup of the model allows us to estimate the dynamics of the default-free term structure separately from the credit-risky dynamics. Before turning to estimation issues, we first specify the specifics of the model below, starting with the setup of the default-free term structure.

2.4 Model: specific setup

Given our focus on the euro area, we need a default-free euro-denominated reference. We take the German term structure as our reference yield curve, and we assume that it is driven by four factors: inflation, the output gap, and two latent factors. We let the default-free (de-meaned) short rate process (which can be viewed as the process of the ECB policy rate) be determined by these same four variables:

$$r_t = \omega_\pi \pi_t + \omega_x x_t + \omega_1 \eta_{1,t} + \eta_{2,t}. \tag{5}$$

This formulation is consistent with typical specifications of monetary policy rules, such as Taylor rules, in structural macro models. In such rules, the short-term policy rate is assumed to react to inflation (possibly in deviation from some implicit or explicit objective of the central bank) and to the output gap. With this type of set-up in mind, the latent factors $\eta_{1,t}$ and $\eta_{2,t}$ could, at least informally, be seen as capturing changes in preferences of the central bank. In the empirical implementation, we will allow $\eta_{1,t}$ to be serially correlated while $\eta_{2,t}$ is assumed to be uncorrelated. As such, these factors could potentially pick up perceived persistent changes to the central bank's inflation objective and transitory "monetary policy shocks", respectively.

The two macro factors and the policy shock, in turn, are assumed to follow a VAR process. We restrict this process in several ways. First, to keep the number of parameters manageable, we allow only one lag (we will be working with a monthly data frequency). The first latent factor is assumed to depend only on its own lag, while the second latent factor is assumed to be a white noise shock. Moreover, both of these factors are allowed to impact on inflation and the output gap (after one lag).

We assume that the credit-specific state variables consist of three variables: a latent credit variable that is common across sovereigns (C_t), the GDP growth rate of country j $\left(g_t^j\right)$, and the fiscal debt/GDP ratio of country j $\left(d_t^j\right)$. Given this, we specify our state vector as (suppressing superscripts j)

$$X_t' = \left[\pi_t, x_t, \eta_{1,t}, \eta_{2,t}, C_t, g_t, d_t\right].$$

The state vector is assumed to follow an AR(1) process, where we impose various restrictions to keep the number of parameters manageable. Specifically, we assume the following structure:

$$\begin{bmatrix} \pi_t \\ x_t \\ \eta_{1,t} \\ \eta_{2,t} \\ C_t \\ g_t \\ d_t \end{bmatrix} = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} & \phi_{1,4} & 0 & 0 & 0 \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} & \phi_{2,4} & 0 & 0 & 0 \\ 0 & 0 & \phi_{3,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{6,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{7,6} & \phi_{7,7} \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ x_{t-1} \\ \eta_{t-1} \\ \eta_{2,t} \\ C_{t-1} \\ g_{t-1} \\ d_{t-1} \end{bmatrix} + \Sigma\varepsilon_t,$$

while Σ is assumed to be diagonal with elements

$$\text{diag}(\Sigma)' = [\sigma_\pi, \sigma_x, 1, 1, 1, \sigma_g, \sigma_d].$$

The risk-free short rate is given by

$$r_t = \delta X_t,$$

with

$$\delta = \left[\omega_\pi, \omega_x, \omega_{\eta_1}, \omega_{\eta_2}, \underset{(1\times 3)}{0}\right],$$

and $\omega_{\eta_1}, \omega_{\eta_2} \geq 0$.

For the specification of the market prices of risk, we impose some restrictions to reduce the number of parameters. We restrict the market price of risk coefficient matrix to take the same form ar the AR coefficient matrix above (but also allwoing for non-zero risk price for $\eta_{2,t-1}$). We then get the following specification:

$$\psi_t = \begin{bmatrix} \psi_0^\pi \\ \psi_0^x \\ \psi_0^{\eta_1} \\ \psi_0^{\eta_2} \\ \psi_0^C \\ \psi_0^g \\ \psi_0^d \end{bmatrix} + \begin{bmatrix} \psi_{\pi,\pi} & \psi_{\pi,x} & \psi_{\pi,\eta_1} & \psi_{\pi,\eta_2} & 0 & 0 & 0 \\ \psi_{x,\pi} & \psi_{x,x} & \psi_{x,\eta_1} & \psi_{x,\eta_2} & 0 & 0 & 0 \\ 0 & 0 & \psi_{\eta_1,\eta_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi_{\eta_1,\eta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{C,C} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{g,g} & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{d,g} & \psi_{d,d} \end{bmatrix} X_t.$$

The risk parameters associated with the euro area macro and monetary policy variables are the same across all countries, while the parameters associated with C, g, and d are allowed to vary across countries.

Finally, we need an assumption regarding the default intensities. In principle, we can allow these to depend directly on the "risk-free" factors, but initially we will assume that they depend only on the credit factors, in the following way:

$$\Lambda_t = \lambda_0 + \lambda_C C_t + \lambda_g g_t + \lambda_d d_t + \lambda_{dd} d_t^2.$$

As a result, Ξ^j in

$$\Lambda_t^j = \lambda_0^j + \lambda^j X_t + X_t' \Xi^j X_t$$

is a matrix of zeros with the lowest right-hand element equal to λ_{dd}.

2.5 Estimation of the reference term structure

We estimate the risk free block in a first step using maximum likelihood, based on the Kalman filter. To construct the likelihood function, we first define a vector W_t containing the observable contemporaneous variables,

$$W_t \equiv \begin{bmatrix} Y_t \\ \pi_t \\ x_t \end{bmatrix},$$

where $\mathbf{Y}_t$ denotes the vector of default-free zero-coupon yields and π_t and x_t are assumed to be de-meaned. The dimension of $\mathbf{W}_t$ is denoted n_w. We can then write the measurement equation as

$$\begin{aligned} W_t &= \begin{bmatrix} A \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} & B' & \\ 1 & 0 & \underset{(2\times 8)}{0} \\ 0 & 1 & \end{bmatrix} X_t + w_t, \\ &\equiv K + H'X_t + w_t, \qquad E\left[\mathbf{w}_t\mathbf{w}_t'\right] = R, \end{aligned}$$

where $\mathbf{w}_t$ is a vector of serially uncorrelated measurement errors corresponding to the observable variables $\mathbf{W}_t$. We assume that, apart from the short-term (one-month) rate, the n_y yields are measured with error (assumed to be cross-sectionally uncorrelated), while the two macro variables have zero measurement error:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{y1}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{y8}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The transition equation is

$$X_t^r = \Phi^r X_{t-1}^r + \mathbf{v}_t^r, \qquad \mathrm{E}\left[\mathbf{v}_t^r \mathbf{v}_t^r\right] = Q, \tag{6}$$

where $Q = \Sigma^r \Sigma^{r\prime}$.

We start the filter from the unconditional mean

$$\mathrm{E}\left[X_t^r\right] = 0,$$

and the unconditional MSE matrix, whose vectorised elements are

$$\operatorname{vec}\left(P_{1|0}\right) = \left(I_{n_u^2} - F \otimes F\right)^{-1} \cdot \operatorname{vec}\left(Q\right),$$

(see Hamilton, 1994).

The Kalman filter will produce forecasts of the states and the associated MSE according to

$$\hat{X}_{t+1|t}^r = \Phi^r \hat{X}_{1,t|t-1}^r + \Phi^r P_{t|t-1} H \left(H' P_{t|t-1} H + R\right)^{-1} \left(W_t - K - H' \hat{X}_{t|t-1}^r\right) \tag{7}$$

$$P_{t+1|t} = \Phi^r P_{t|t-1} \Phi^{r\prime} - \Phi^r P_{t|t-1} H \left(H' P_{t|t-1} H + R\right)^{-1} H' P_{t|t-1} \Phi^{r\prime} + Q. \tag{8}$$

Given this, the likelihood can be expressed as

$$\begin{aligned} \sum_{t=1}^{T} \log f\left(W_t \mid W_{t-1}, \theta_0\right) = &-\frac{T\left(n_y - 1\right)}{2} \ln\left(2\pi\right) - \frac{1}{2} \sum_{t=1}^{T} \ln\left|\Sigma_t\left[\theta_0\right]\right| \\ &- \frac{1}{2} \sum_{t=1}^{T} \left(W_t - \mu_t\left[\theta_0\right]\right)' \left(\Sigma_t\left[\theta_0\right]\right)^{-1} \left(W_t - \mu_t\left[\theta_0\right]\right) \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Sigma_t\left[\theta_0\right] &\equiv H\left[\theta_0\right]' P_{t|t-1}\left[\theta_0\right] H\left[\theta_0\right] + R\left[\theta_0\right], \\ \mu_t\left[\theta_0\right] &\equiv K\left[\theta_0\right] + H\left[\theta_0\right]' \hat{X}_{t|t-1}^r\left[\theta_0\right]. \end{aligned}$$

Given that the parameter space is quite large, we employ the method of simulated annealing, introduced to the econometric literature by Goffe, Ferrier and Rogers (1994). The method is developed with an aim towards applications where there may be a large number of local optima.

2.6 Estimation of credit risky term structures

In our setup, yields on credit risky bonds are non-linear functions of the state variables. As a result, we can no longer use the standard Kalman filter to estimate the model. [discussion of possible alternatives] Instead, we rely on the unscented Kalman filter of Julier and Uhlmann (1997, 2004). The unscented Kalman filter relies on a deterministic sampling technique to pick points around the mean of some underlying random variable. These so-called sigma points are then propagated through the non-linear functions of interest, in order to recover the first two moments of the non-linear system. These can subsequently be used in the updating step of the filter.

In our application, the transition equation is

$$X_t = \Phi X_{t-1} + \Sigma \varepsilon_t, \tag{10}$$

while the observation equation is

$$z_t = \Theta\left(X_t\right) + \xi_t, \tag{11}$$

where z_t is a vector of observables (e.g. yields on risky bonds), $\Theta\left(\cdot\right)$ is a non-linear function, and where the observation error vector ξ_t is assumed to have zero mean and a diagonal covariance matrix $\tilde{R}$. Similar to the standard Kalman filter, the unscented filter relies on a linear updating rule according to

$$\hat{X}_{t|t} = \hat{X}_{t|t-1} + \tilde{K}_t\left(z_t - \hat{z}_{t|t-1}\right), \tag{12}$$

where

$$\begin{aligned} \hat{X}_{t|t-1} &= \Phi \hat{X}_{t-1|t-1}, \\ \tilde{K}_t &= P_{xz(t|t-1)} P^{-1}_{zz(t|t-1)}, \\ \hat{z}_{t|t-1} &= \mathrm{E}\left[\Theta\left(\hat{X}_{t|t-1}\right)\right], \end{aligned}$$

and where P_{zz} is the innovation covariance matrix and P_{xz} is the cross covariance matrix. The updated state is associated with updated covariance

$$P_{xx(t|t)} = P_{xx(t|t-1)} - \tilde{K}_t P_{xx(t|t-1)} \tilde{K}'_t, \tag{13}$$

where

$$P_{xx(t|t-1)} = \Phi P_{xx(t-1|t-1)} \Phi' + \Sigma\Sigma'.$$

For an n_x-dimensional state vector X, a set of $2n_x + 1$ sigma points $\varkappa_0$, $\varkappa_1, ...,$ $\varkappa_{n_x}$ with associated weights ϖ_0, $\varpi_1, ...,$ ϖ_{n_x} are chosen (see the Appendix [to come] for details). For each sigma point i, the nonlinear transformation in (11) is applied

$$\mathcal{Z}_i = \Theta\left(\varkappa_i\right).$$

The covariance matrices P_{xx}, P_{zz} and P_{xz} are then approximated using $\varkappa_i$ and the transformed points $\mathcal{Z}_i$ (see Appendix [to come]).

In our case, the observation vector consists of n_s risky zero-coupon bond yields for each country j, stacked in s_t^j, and a vector f_t^j that contains data on the expected fiscal position and expected GDP growth rate of country j, based on forecasts of the deficit to GDP ratio and GDP growth (the exact nature of the data is discussed below). Given data for m countries, we can define the observation vector as

$$z_t \equiv \begin{bmatrix} \mathbf{s}_t^1 \\ \vdots \\ \mathbf{s}_t^m \\ f_t^1 \\ \vdots \\ f_t^m \end{bmatrix}.$$

Note that we keep the systematic state variables of the risk-free block that we obtained in the first estimation step fixed at their filtered values when performing the second step.

3 Data

Our data is monthly and covers the period from the introduction of the euro, January 1999 to December 2010 [soon to be updated to end-2011]. The default-free term structure data consists of German zero-coupon yields with maturities 1, 3, 6 months, and 1, 3, 5, 7, and 10 years, which are estimated by the German Bundesbank. To model the perceived behaviour of the ECB, we use euro area aggregate values of inflation and the output gap. The measure of inflation is monthly *y-o-y* HICP log-differences. For the output gap, we follow Clarida, Galí and Gertler (1998) and measure the output gap as deviations of real GDP from a quadratic trend. To obtain a monthly series for the gap, we fit an ARMA(1,1) model to the quarterly gap series, then forecast the gap one quarter ahead and compute one- and two-month ahead values by means of linear interpolation. This exercise is conducted in "real time", in the sense that the model is reestimated at each quarter using data only up to that quarter.

For the estimation of the credit risky term structures, we use 2, 5, and 10-year yields for Greece, Portugal, Spain, France and Italy. We estimate zero-coupon yields for these countries based on prices of all available government bonds at each point in time, as reported by Bloomberg, using the Nelson-Siegel model. We use data on zero-coupon bonds with maturities 2, 3, 4, 5, 7 and 10 years when estimating the model.

Finally, we include forecast data on the deficit/GDP ratios and GDP growth figures for these five countries. Twice a year, the European Commission provides data on such forecasts for horizons roughly one and two years ahead. For the forecast released in the Spring, the forecasts cover the current and next year, i.e. until the end of the current year and until the end of the following year. For the Fall forecast, the horizons extend through the next and the following years. By including this data, we are implicitly assuming that investors make similar deficit forecasts as the European Commission, or that they take these forecasts into account in their pricing decisions when they are made public. By using forecasts rather than official data on deficits, we hope to be able to capture some of the forward-lookingness of financial markets.

4 Estimates

We first discuss the estimates of the benchmark German term structure, which we take to be free of default risk. We then go on to look at the estimates for the four credit risky countries included in our sample.

4.1 Estimates for Germany

The first step of our estimation procedure gives us the dynamics of the German term structure, including the implied interest rate setting behaviour of the ECB. For the default-free short rate, we obtain the following estimates,

$$r_t = 0.97\pi_t + 0.32x_t + 0.26\eta_{1,t} + 0.76\eta_{2,t},$$

with state variable dynamics

$$\begin{bmatrix} \pi_t \\ x_t \\ \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} = \begin{bmatrix} 0.84 & 0.10 & 0.01 & 0.49 \\ -0.12 & 0.99 & 0.00 & 0.01 \\ 0 & 0 & 0.98 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ x_{t-1} \\ \eta_{t-1} \\ \eta_{2,t} \end{bmatrix} + \Sigma\varepsilon_t,$$

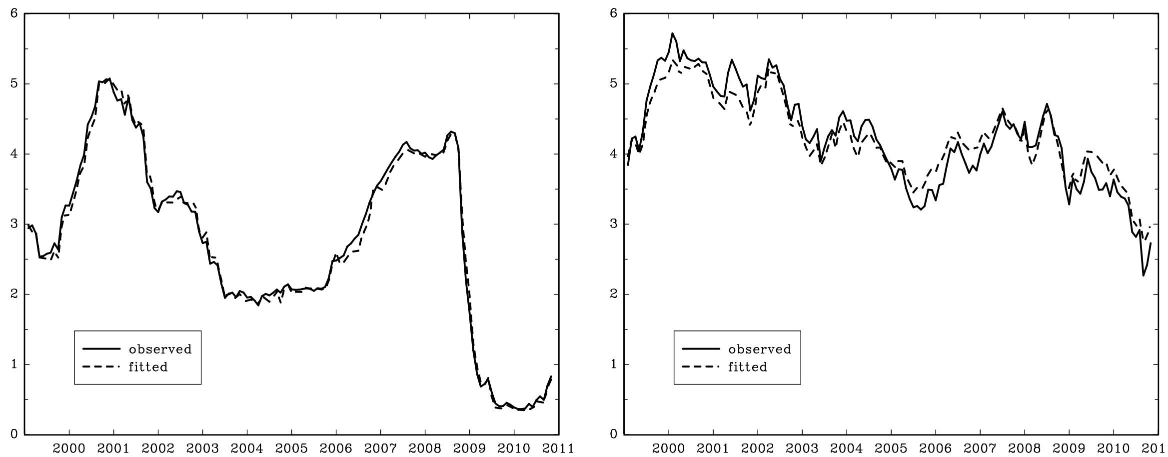
with

$$\Sigma = \begin{bmatrix} 0.59 & 0 & 0 & 0 \\ 0 & 0.29 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[standard errors to be added].

As displayed in Figure 2, the fit of the model is quite good in terms of how well it can capture the dynamics of observed German yields. The standard deviations of the yield measurement errors range from 5 to 21 basis points per year for the seven imperfectly observed yield series. This is in line with typical results reported in the empirical term structure literature (e.g. Dai and Singleton (2000) and Duffee (2002)).

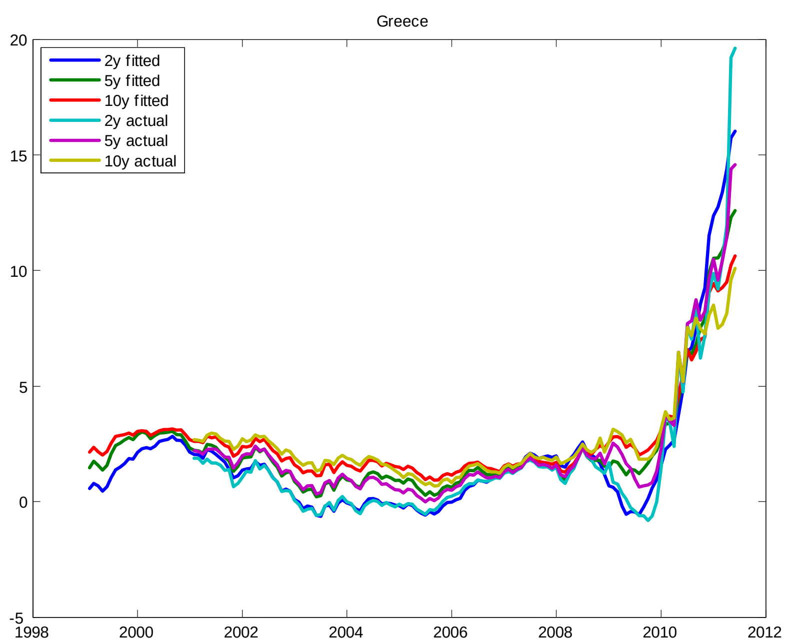
Figure 2: Actual and estimated German yields: 3-month maturity (left) and 10-year maturity (right)

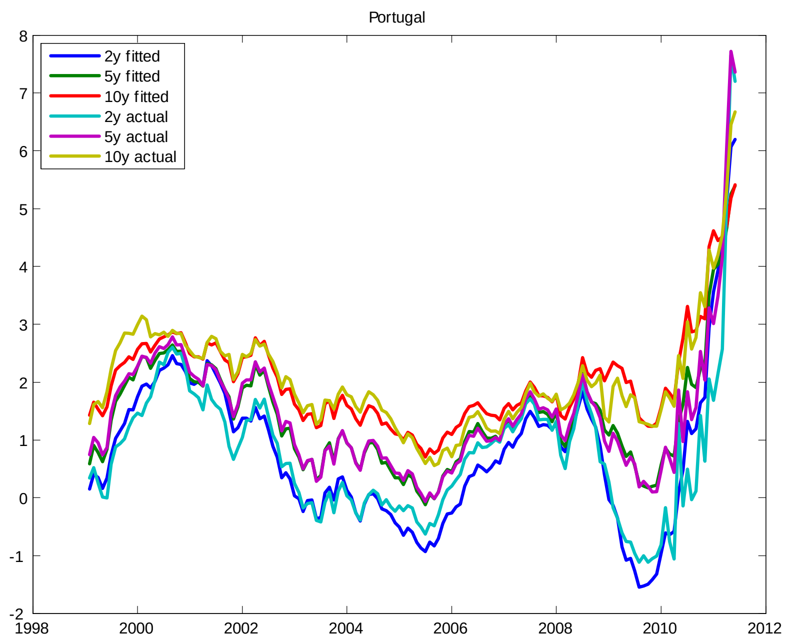


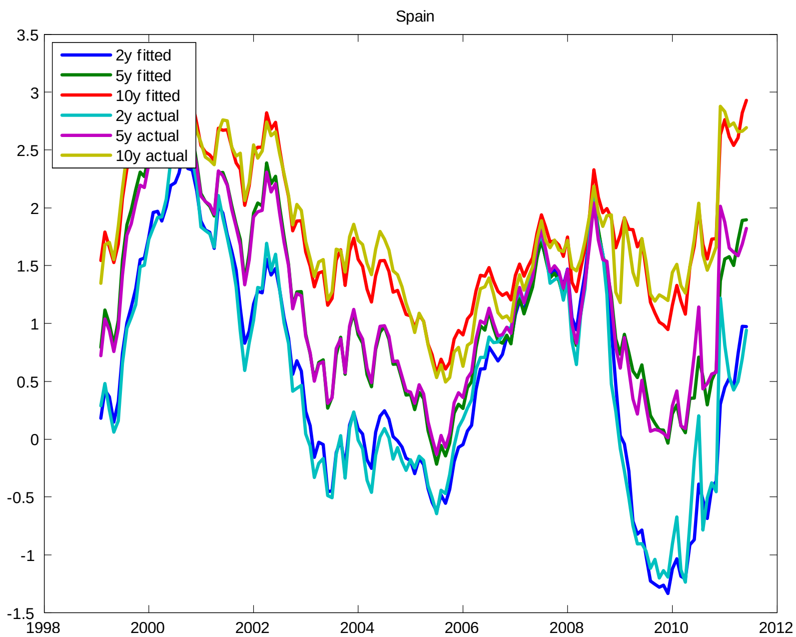
4.2 Estimates for Greece, Portugal, Spain, France and Italy

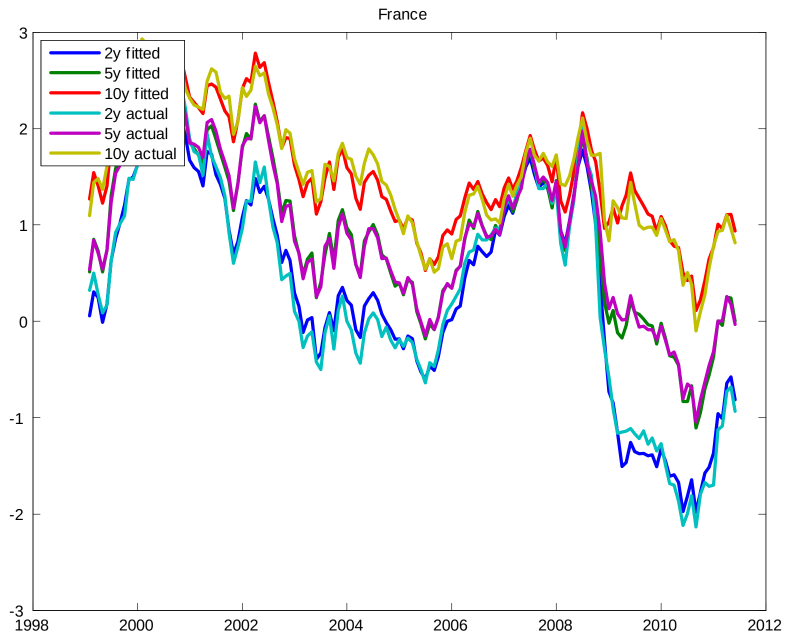
[text to come]

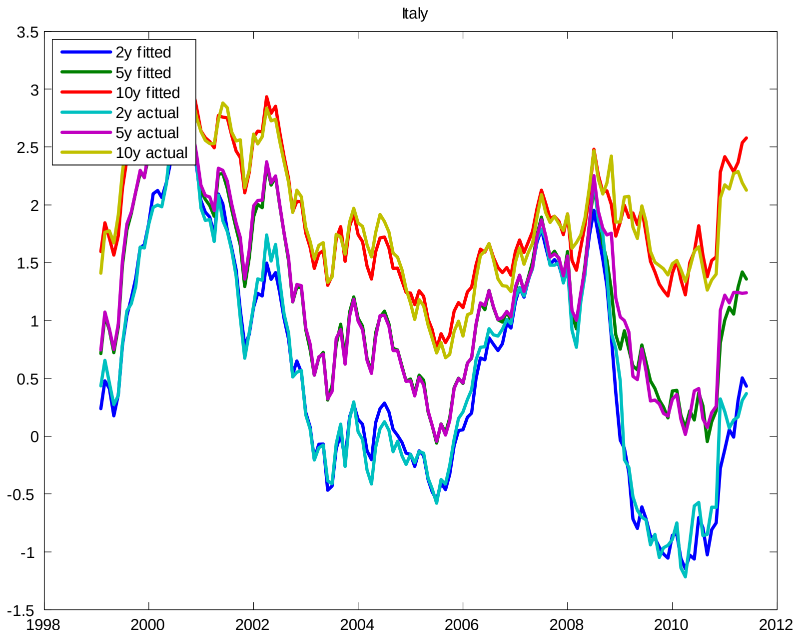
Model fit:



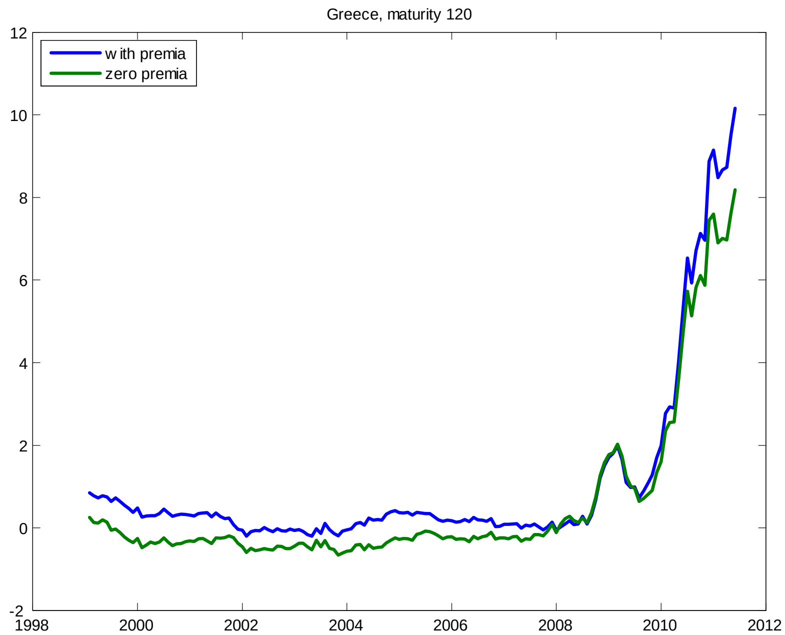

Portugal
2y fitted
5y fitted
10y fitted
2y actual
5y actual
10y actual
8
7
6
5
4
3
2
1
0
-1
-2
1998
2000
2002
2004
2006
2008
2010
2012

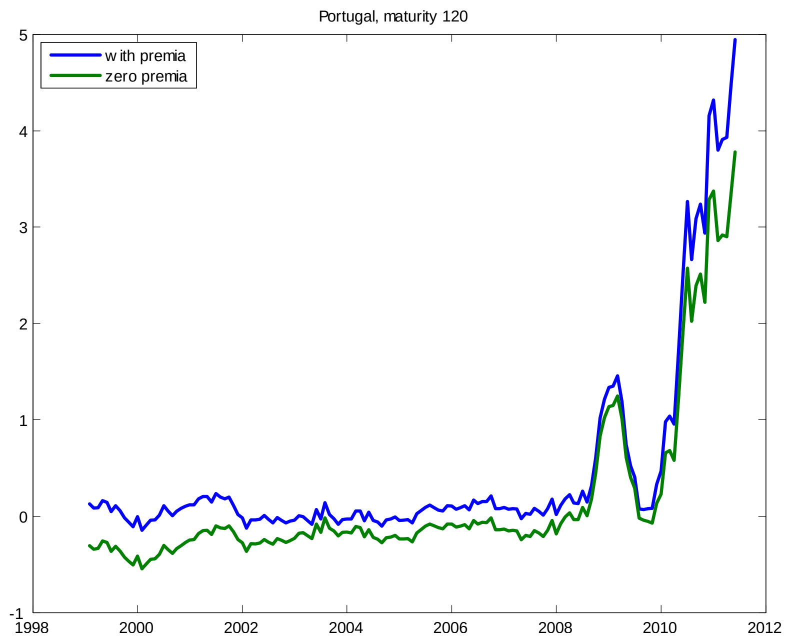

Spain
2y fitted
5y fitted
10y fitted
2y actual
5y actual
10y actual
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
1998
2000
2002
2004
2006
2008
2010
2012

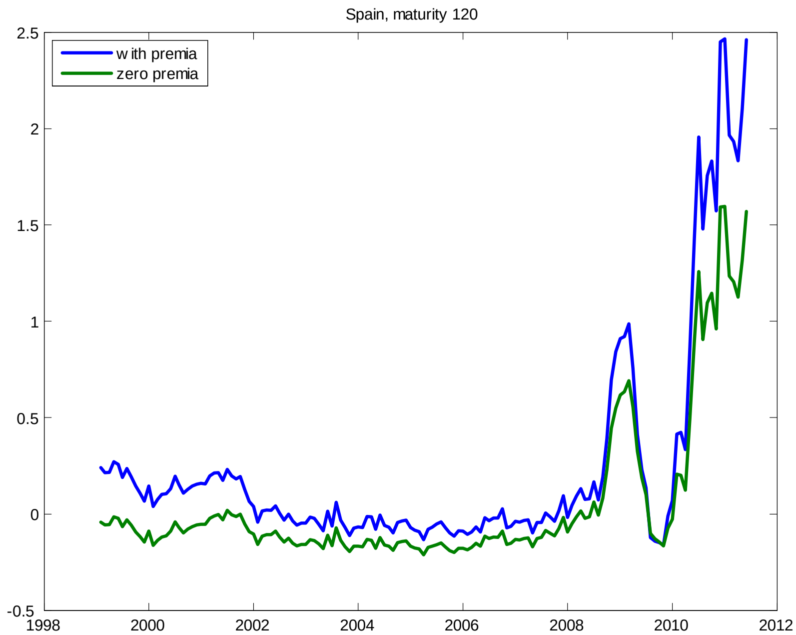


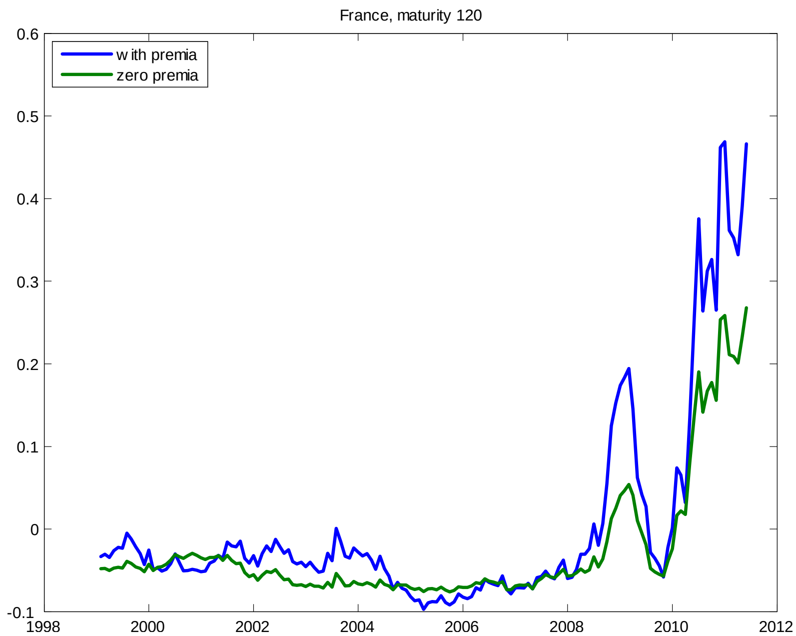


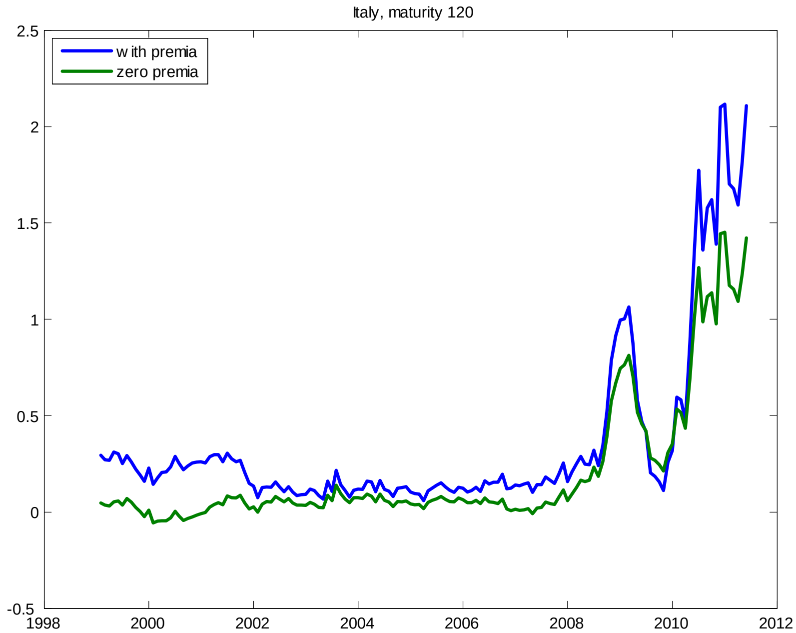
Credit risk premium / default risk decompositions:


Greece, maturity 120
with premia
zero premia
12
10
8
6
4
2
0
-2
1998
2000
2002
2004
2006
2008
2010
2012


Portugal, maturity 120
with premia
zero premia
5
4
3
2
1
0
-1
1998
2000
2002
2004
2006
2008
2010
2012


Spain, maturity 120
w ith premia
zero premia
2.5
2
1.5
1
0.5
0
-0.5
1998
2000
2002
2004
2006
2008
2010
2012


France, maturity 120
w ith premia
zero premia
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
1998
2000
2002
2004
2006
2008
2010
2012



5 Conclusions

[to come]

A Appendix:

A.1 Credit-risky bond prices

The price at t of a risky bond maturing at $t+n$ can be written as

$$B_t^{t+n} = E_t \left[m_{t,t+1} \left(B_{t+1}^{t+n} \mathbf{1}_{\tau > t+1} + Z_{t+1} \mathbf{1}_{\tau < t+1} \right) \right],$$

with boundary condition

$$B_{t+n-1}^{t+n} = E_{t+n-1} \left[m_{t+n-1,t+n} \left(B_{t+n-1}^{t+n} \mathbf{1}_{\tau > t+n} + Z_{t+n} \mathbf{1}_{\tau < t+n} \right) \right],$$

where Z_t is the recovery payment, τ denotes the time of default and $\mathbf{1}_{\tau > t+1}$ is an indicator variable that takes the value one if $\tau > t+1$. In general, the expectation $E_t \left[\mathbf{1}_{\tau > t+k} \right]$ is the probability of survival until $t+k$:

$$E_t \left[\mathbf{1}_{\tau > t+k} \right] = E_t \left[\exp \left(- \sum_{i=1}^{k} \Lambda_{t+i} \right) \right].$$

Under a RMV assumption, the expected recovery payment is a fraction of the bond price at $t+1$, conditional on no default, ie for an n-period bond

$$E_t \left[Z_{t+1} \right] = E_t \left[\left(1 - L_{t+1} \right) B_{t+1}^{t+n} \right],$$

where L_{t+1} is the fractional loss rate.

Assuming that the loss rate is a constant L, we have, under RMV,

$$B_t^{t+n} = E_t \left[m_{t,t+1} \left(B_{t+1}^{t+n} \mathbf{1}_{\tau > t+1} + (1 - L) B_{t+1}^{t+n} \mathbf{1}_{\tau < t+1} \right) \right],$$

which can be written

$$\begin{aligned} B_t^{t+n} &= E_t \left[m_{t,t+1} \left(B_{t+1}^{t+n} \exp(-\Lambda_{t+1}) + B_{t+1}^{t+n} (1 - L) (1 - \exp(-\Lambda_{t+1})) \right) \right] \\ &= E_t \left[m_{t,t+1} \left(\exp(-\Lambda_{t+1}) + (1 - L)(1 - \exp(-\Lambda_{t+1})) \right) B_{t+1}^{t+n} \right] \\ &= E_t \left[m_{t,t+1} \left(1 - L(1 - \exp(-\Lambda_{t+1})) \right) B_{t+1}^{t+n} \right]. \end{aligned}$$

Assume that we can make the following approximation

$$1 - L(1 - \exp(-\Lambda_{t+1})) \approx \exp(-\Lambda_{t+1}).$$

This approximation holds exactly for $L = 1$. For L different from 1, we should view Λ as reflecting adjusted default intensities, rather than actual intensities. This analogous to the use of "recovery-adjusted default intensities" in continuous time models with RMV (e.g. Duffie and Singleton, 1999). Given this assumption, we can write

$$B_t^{t+n} = E_t \left[m_{t,t+1} \exp(-\Lambda_{t+1}) B_{t+1}^{t+n} \right].$$

We will assume that the (adjusted) default intensity of country j is a quadratic function of the states:

$$\Lambda_t^j = \lambda_0^j + \lambda^j X_t + X_t' \Xi^j X_t.$$

The price of an n-period bond is therefore (supressing superscripts j)

$$\begin{aligned}
B_t^{t+n} &= E_t\left[m_{t,t+1}\exp\left(-\Lambda_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= E_t\left[\exp\left(-r_t-\frac{1}{2}\psi_t'\psi_t-\psi_t'\varepsilon_{t+1}\right)\exp\left(-\lambda_0-\lambda X_{t+1}-X_{t+1}'\Xi X_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= E_t\left[\exp\left(\begin{array}{c}-\delta_0-\delta X_t-\frac{1}{2}\left(\psi_0'\psi_0+2\psi_0'\psi_1X_t+X_t'\psi_1'\psi_1X_t\right)-\left(\psi_0'+X_t'\psi_1'\right)\varepsilon_{t+1} \\ -\lambda_0-\lambda\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)-\left(X_t'\Phi'+\varepsilon_{t+1}'\Sigma'\right)\Xi\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)\end{array}\right)B_{t+1}^{t+n}\right] \\
&= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t\right) \\
&\quad\times E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma\right)\varepsilon_{t+1}-X_t'\Phi'\Xi\Phi X_t-2X_t'\Phi'\Xi\Sigma\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t-X_t'\Phi'\Xi\Phi X_t\right) \\
&\quad\times E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)B_{t+1}^{t+n}\right]
\end{aligned}$$

We know that we can write the price of a bond as

$$B_t^{t+n}=\exp\left(A_n+B_nX_t+X_t'C_nX_t\right).$$

We plug in

$$\begin{aligned}
B_{t+1}^{t+n} &= \exp\left(A_{n-1}+B_{n-1}X_{t+1}+X_{t+1}'C_{n-1}X_{t+1}\right) \\
&= \exp\left(A_{n-1}+B_{n-1}\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)+\left(X_t'\Phi'+\varepsilon_{t+1}'\Sigma'\right)C_{n-1}\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)\right) \\
&= \exp\left(\begin{array}{c}A_{n-1}+B_{n-1}\Phi X_t+B_{n-1}\Sigma\varepsilon_{t+1} \\ +X_t'\Phi'C_{n-1}\Phi X_t+2X_t'\Phi'C_{n-1}\Sigma\varepsilon_{t+1}+\varepsilon_{t+1}'\Sigma'C_{n-1}\Sigma\varepsilon_{t+1}\end{array}\right)
\end{aligned}$$

into the bond price above to get

$$\begin{aligned}
B_t^{t+n} &= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t-X_t'\Phi'\Xi\Phi X_t\right) \\
&\quad\times E_t\left[\exp\left(\begin{array}{c}-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1} \\ +A_{n-1}+B_{n-1}\Phi X_t+B_{n-1}\Sigma\varepsilon_{t+1} \\ +X_t'\Phi'C_{n-1}\Phi X_t+2X_t'\Phi'C_{n-1}\Sigma\varepsilon_{t+1}+\varepsilon_{t+1}'\Sigma'C_{n-1}\Sigma\varepsilon_{t+1}\end{array}\right)\right] \\
&= \exp\left(\begin{array}{c}-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0+A_{n-1}+\left(B_{n-1}\Phi-\delta-\lambda\Phi-\psi_0'\psi_1\right)X_t \\ +X_t'\left(\Phi'C_{n-1}\Phi-\Phi'\Xi\Phi-\frac{1}{2}\psi_1'\psi_1\right)X_t\end{array}\right) \\
&\quad\times E_t\left[\exp\left(\begin{array}{c}\left(B_{n-1}\Sigma+2X_t'\Phi'C_{n-1}\Sigma-\psi_0'-X_t'\psi_1'-\lambda\Sigma-2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right].
\end{aligned}$$

Rewrite the expectation as

$$\begin{aligned}
&E_t\left[\exp\left(\begin{array}{c}\left(B_{n-1}\Sigma+2X_t'\Phi'C_{n-1}\Sigma-\psi_0'-X_t'\psi_1'-\lambda\Sigma-2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right] \\
=\; &E_t\left[\exp\left(\begin{array}{c}\left(B_{n-1}+2X_t'\Phi'C_{n-1}-\psi_0'\Sigma^{-1}-X_t'\psi_1'\Sigma^{-1}-\lambda-2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right] \\
=\; &E_t\left[\exp\left(aw_{t+1}+w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right],
\end{aligned}$$

where

$$
\begin{aligned}
w_{t+1} &\equiv \Sigma\varepsilon_{t+1} \\
a &\equiv B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi, \\
\bar{C}_{n-1} &\equiv C_{n-1} - \Xi.
\end{aligned}
$$

To evaluate the expectation we follow Realdon (2006), who demonstrates that (if γ is of full rank)

$$
E_t\left[\exp\left(aw_{t+1} + w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right] = \frac{|\gamma|}{\text{abs}\,|\Sigma|}\prod_{i=1}^{N}\exp\left(\frac{(a\gamma_i)^2}{2}\right)
$$

where $\gamma \equiv \left((\Sigma\Sigma')^{-1} - 2\bar{C}_{n-1}\right)^{-1/2}$, γ_i denotes the i-th column of γ, $|\gamma|$ denotes the determinant of γ and abs$|\Sigma|$ denotes the absolute value of the determinant of Σ. We therefore get

$$
\begin{aligned}
& E_t\left[\exp\left(aw_{t+1} + w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right] \\
= & \frac{|\gamma|}{\text{abs}\,|\Sigma|}\prod_{i=1}^{N}\exp\left(\frac{\left(\left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i\right)^2}{2}\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
\ln B_t^{t+n} = & -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + A_{n-1} + \left(B_{n-1}\Phi - \delta - \lambda\Phi - \psi_0'\psi_1\right)X_t \\
& + X_t'\left(\Phi' C_{n-1}\Phi - \Phi'\Xi\Phi - \frac{1}{2}\psi_1'\psi_1\right)X_t \\
& + \ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i\right)^2.
\end{aligned}
$$

Evaluating the squared term:

$$
\begin{aligned}
& \left(\left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i\right)^2 \\
= & \left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i \\
& \times\gamma_i'\left(B_{n-1}' + 2C_{n-1}'\Phi X_t - \Sigma^{-1'}\psi_0 - \Sigma^{-1'}\psi_1 X_t - \lambda' - 2\Xi'\Phi X_t\right) \\
= & \left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(B_{n-1}' - \Sigma^{-1'}\psi_0 - \lambda'\right) \\
& + 2\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t \\
& + X_t'\left(2\Phi' C_{n-1} - \psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t,
\end{aligned}
$$

we get

$$
\begin{aligned}
\ln B_t^{t+n} \;=\; & -\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0+A_{n-1}+\left(B_{n-1}\Phi-\delta-\lambda\Phi-\psi_0'\psi_1\right)X_t \\
& +X_t'\left(\Phi'C_{n-1}\Phi-\Phi'\Xi\Phi-\frac{1}{2}\psi_1'\psi_1\right)X_t \\
& +\ln\frac{|\gamma|}{\text{abs}\,|\Sigma|}+\frac{1}{2}\sum_{i=1}^{N}\left(B_{n-1}-\psi_0'\Sigma^{-1}-\lambda\right)\gamma_i\gamma_i'\left(B_{n-1}'-\Sigma^{-1'}\psi_0-\lambda'\right) \\
& +\sum_{i=1}^{N}\left(B_{n-1}-\psi_0'\Sigma^{-1}-\lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi-\Sigma^{-1'}\psi_1-2\Xi'\Phi\right)X_t \\
& +\frac{1}{2}\sum_{i=1}^{N}X_t'\left(2\Phi'C_{n-1}-\psi_1'\Sigma^{-1}-2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi-\Sigma^{-1'}\psi_1-2\Xi'\Phi\right)X_t.
\end{aligned}
$$

We can therefore identify the recusive factor loadings of the bond price $B_t^{t+n} = \exp\left(A_n+B_nX_t+X_t'C_nX_t\right)$ as

$$
\begin{aligned}
A_n \;=\; & A_{n-1}-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0+\ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} \\
& +\frac{1}{2}\sum_{i=1}^{N}\left(B_{n-1}-\psi_0'\Sigma^{-1}-\lambda\right)\gamma_i\gamma_i'\left(B_{n-1}'-\Sigma^{-1'}\psi_0-\lambda'\right), \\
B_n \;=\; & B_{n-1}\Phi-\delta-\lambda\Phi-\psi_0'\psi_1 \\
& +\sum_{i=1}^{N}\left(B_{n-1}-\psi_0'\Sigma^{-1}-\lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi-\Sigma^{-1'}\psi_1-2\Xi'\Phi\right), \\
C_n \;=\; & \Phi'C_{n-1}\Phi-\Phi'\Xi\Phi-\frac{1}{2}\psi_1'\psi_1 \\
& +\frac{1}{2}\sum_{i=1}^{N}\left(2\Phi'C_{n-1}-\psi_1'\Sigma^{-1}-2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi-\Sigma^{-1'}\psi_1-2\Xi'\Phi\right).
\end{aligned}
$$

To get the initial conditions, consider the price of a 1-period bond:

$$
\begin{aligned}
B_t^{t+1} \;=\; & \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t-X_t'\Phi'\Xi\Phi X_t\right) \\
& \times E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right].
\end{aligned}
$$

Rewrite the expectation as

$$
\begin{aligned}
& E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right] \\
= \;& E_t\left[\exp\left(\left(-\psi_0'\Sigma^{-1}-X_t'\psi_1'\Sigma^{-1}-\lambda-2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right] \\
= \;& E_t\left[\exp\left(a_1w_{t+1}-w_{t+1}'\Xi w_{t+1}\right)\right],
\end{aligned}
$$

where

$$
a_1\equiv-\psi_0'\Sigma^{-1}-X_t'\psi_1'\Sigma^{-1}-\lambda-2X_t'\Phi'\Xi,
$$

so that

$$E_t\left[\exp\left(a_1 w_{t+1} - w_{t+1}'\Xi w_{t+1}\right)\right] = \frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|}\prod_{i=1}^{N}\exp\left(\frac{\left(a_1\gamma_i^1\right)^2}{2}\right)$$

where $\gamma^1 \equiv \left(\left(\Sigma\Sigma'\right)^{-1} + 2\Xi\right)^{-1/2}$. We get

$$\begin{aligned} B_t^{t+1} &= \exp\left(-\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \delta X_t - \lambda\Phi X_t - \psi_0'\psi_1 X_t - \frac{1}{2}X_t'\psi_1'\psi_1 X_t - X_t'\Phi'\Xi\Phi X_t\right) \\ &\quad \times E_t\left[\exp\left(-\left(\psi_0' + X_t'\psi_1' + \lambda\Sigma + 2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} - \varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right]. \end{aligned}$$

$$\begin{aligned} \ln B_t^{t+1} &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \left(\delta + \lambda\Phi + \psi_0'\psi_1\right)X_t - X_t'\left(\frac{1}{2}\psi_1'\psi_1 + \Phi'\Xi\Phi\right)X_t \\ &\quad + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i^1\right)^2. \end{aligned}$$

The squared term is:

$$\begin{aligned} & \left(\left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i^1\right)^2 \\ = & \left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i^1 \\ & \times\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \Sigma^{-1'}\psi_1 X_t - \lambda' - 2\Xi'\Phi X_t\right) \\ = & \left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \lambda'\right) \\ & +2\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t \\ & +X_t'\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t, \end{aligned}$$

so that

$$\begin{aligned} \ln B_t^{t+1} &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \left(\delta + \lambda\Phi + \psi_0'\psi_1\right)X_t - X_t'\left(\frac{1}{2}\psi_1'\psi_1 + \Phi'\Xi\Phi\right)X_t \\ &\quad + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \lambda'\right) \\ &\quad + \sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}X_t'\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t, \end{aligned}$$

which gives

$$
\begin{aligned}
A_1 &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + \ln\frac{\left|\gamma^1\right|}{\mathrm{abs}\left|\Sigma\right|} + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \lambda'\right), \\
B_1 &= -\delta - \lambda\Phi - \psi_0'\psi_1 + \sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right) \\
C_1 &= -\frac{1}{2}\psi_1'\psi_1 - \Phi'\Xi\Phi + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right).
\end{aligned}
$$

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