Risky Linear Approximations*

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Abstract

I construct risk-corrected approximations around the stochastic steady state and ergodic mean that are linear in the state variables. The resulting approximations are uniformly more accurate than standard linear approximations and capture the dynamics of asset pricing variables such as the expected risk premium missed by standard linear approximations. The algorithm is fast and reliable, requiring only the solution of linear equations using standard perturbation output. I examine the joint macroeconomic and asset pricing implications of a real business cycle model with stochastic trends and recursive preferences. The method is able to estimate risk aversion under these preferences using the Kalman filter, where a standard linear approximation provides no information and alternative methods require computationally intensive particle filters subject to sampling variation.

JEL classification: C61, C63, E17

Keywords: DSGE; Solution methods; Ergodic mean; Stochastic steady state; Perturbation

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1 Introduction

What are the effects of small shock realizations in economies subject to risk? There is a growing consensus that standard linear approximations around the deterministic steady state are insufficient to address this question satisfactorily in a variety of settings, such as for conditional asset pricing, under recursive utility, for welfare comparisons, etc., where precautionary motives play a significant role in economic decision making. The reason is simple, standard linear approximations around the deterministic steady state are certainty equivalent, that is, they are impervious to risk, the moments of the distribution of exogenous shocks.[1]

I reconcile the linear framework with risk by constructing approximations that are linear in states but that account for risk in the points and slopes used to construct the linear approximation; I call these risky linear approximations.[2] I construct two different approximations, one around the stochastic steady state and one around the ergodic mean. The method can be used profitably in estimation. Due to the linearity in states and under the assumption of normally distributed shocks, the Kalman filter is operational for the risky linear approximation. I find the risky linear Kalman filter approximation is as equally successful as standard perturbation particle filter[3] estimation—both state space and nonlinear moving average[4]—in identifying parameters outside the reach of standard linear approximations. The advantage, then, is that the risky linear approximation, by employing the Kalman filter, is several orders of magnitude faster and is not subject to the sampling variation that the parti-

[1]Kim and Kim (2003) provide an insightful example, where blind application of linear approximations leads to the spurious results that autarky is to be preferred over risk sharing given risk averse utility.

[2]I construct only risky linear approximations and not second, third, or higher order risky approximations as standard DSGE perturbations, see, e.g., Jin and Judd (2002), provide the appropriate, in the sense of Taylor's theorem, local polynomial approximation fo the policy function using derivatives calculated at the deterministic steady state. The risky linear approximation I derive uses some derivatives from a given order and discards others. This price is worth paying for maintaining linearity in states, which resuscitates the linear toolbox of DSGE analysis including the Kalman filter for estimation, while incorporating the nonlinearities associated with risk. It is less clear what the gain over a standard perturbation would be for higher order risky approximations.

[3]See Fernández-Villaverde and Rubio-Ramírez (2007) and Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011) for details on particle filtering in DSGE models as well as applications to risk.

[4]See Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Kim, Kim, Schaumburg, and Sims (2008), Lombardo (2010) and Lan and Meyer-Gohde (2013c)

cle filter faces when identification is weak.

I apply the method to a real business cycle model with risk sensitive, using Epstein and Zin (1989) and Weil (1990) recursive preferences, and long run risk following Bansal and Yaron (2004). The risky linear approximations match the stochastic steady state, ergodic mean, and impulse responses reported in previous nonlinear studies.[5] I assess the accuracy of the resulting linear in state variables approximation using Euler equation errors and find that the risky linear approximation displays a uniform improvement over the standard linear approximation. As risk plays a greater role—i.e., risk aversion is increased—the risky linear approximation demonstrates accuracy in the vicinity of the stochastic steady state and ergodic mean that is comparable to second and third order perturbations. I find that US post war data on consumption, output, and the risk premium leads the likelihood function to favor higher levels of risk aversion with the posterior mode at about 30. The likelihood function, however, is rather flat in this dimension and fully nonlinear approximations employing the particle filter suffer from sampling variation that impedes reliable inference; for standard linear approximations, the likelihood function is entirely flat in the dimension of risk aversion and the posterior is identical to the prior.

The method I propose here is not the first method to employ a risk correction as opposed to a standard nonlinear method (higher order perturbation or global solution method such as projection),[6] but it is different in two important respects. First, it is the first method to work solely with derivative information from a standard perturbation and known moments of the exogenous process—no reevaluation of derivatives or recalculation of policy rules are required to construct the approximation. The

[5]The stochastic steady state derived here is identical to that of Lan and Meyer-Gohde (2013b), the ergodic mean identical to that of Lan and Meyer-Gohde (2013a) and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013), and the impulse responses to those that would result from the method of Lan and Meyer-Gohde (2013c)

[6]Kim and Kim (2003) as well as Collard and Juillard (2001b) and Collard and Juillard (2001a) are early DSGE bias reduction or risk correction techniques. Coeurdacier, Rey, and Winant (2011) uses a second order approximation to the equilibrium conditions to solve for the stochastic steady state in a portfolio problem, de Groot (2013) extends this to general settings as a matrix quadratic problem. Juillard (2011) and Kliem and Uhlig (2013) use iterative techniques, solving for implied stochastic steady states given an approximated solution and then recalculating the approximation at the new implied stochastic steady state. Evers (2012) solves for the stochastic steady state implied by a risk perturbation of the equilibrium conditions and then solves for a perturbation in the states of these perturbed equilibrium conditions.

resulting equations are linear in the unknown coefficients of my approximation entirely avoiding fix point or other recursive algorithms with unknown convergence properties. Second, my approximation allows for the approximation around the ergodic mean as well as around the stochastic steady state—competing methods can provide only the latter. Both of these two features are accomplished by working implicitly with the unknown policy function instead of the model equilibrium conditions.

The remainder of the paper is organized as follows. In section 2, I lay out the model class and assumptions underlying the local, risk corrected procedure behind risky linear approximation. I proceed to derive the risky linear approximations in section 3. In section 4, I derive the real business cycle model with recursive preferences and long run risk. I analyze two calibrated versions of the real business cycle model in terms of accuracy in section 5. In section 6, I assess the likelihood properties of the risky linear approximation relative to particle filters and standard linearizations and estimate risk aversion and long run risk using post war US data in section 6. Section 7 concludes.

2 DSGE Model: Assumptions and Local Approximation

I begin by introducing a general class of models, a system of nonlinear second order expectational difference equations.

2.1 Model Class

I will analyze a family of discrete-time rational expectations models given by

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \sigma\varepsilon_t)] \quad (1)$$

$f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_y}$ is a n_y-dimensional vector-valued function collecting the equilibrium conditions that describe the model; $y_t \in \mathbb{R}^{n_y}$ is the vector of n_y endogenous and exogenous variables;[7] and $\varepsilon_t \in \mathbb{R}^{n_e}$ the vector of n_e exogenous shocks,[8] where n_y and n_e are positive integers ($n_y, n_e \in \mathbb{N}$).

[7] As well as subsidiary definitions to bring a model into the form of (1). For the model of the paper here, this requires the definition of $U_t^e \doteq E_t[U_{t+1}]$ and $M_t^e \doteq E_t[M_{t+1}]$, to capture the nonlinearities over the conditional expectations while conforming to the class in (1).

[8] This model class encompasses competitive equilibria and dynamic programming problems, as well as models with finitely many heterogenous agents, see Judd and Mertens (2012). Nonlinearity or serial correlation in exogenous pro-

The auxiliary parameter $\sigma \in \mathbb{R}$ scales the risk in the model.[9] The stochastic model version of (1) corresponds to $\sigma = 1$ and $\sigma = 0$ represents the deterministic version of the model. Indexing solutions by σ yields

$$y_t = g(y_{t-1}, \sigma\varepsilon_t, \sigma), \quad y : \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \times \mathbb{R} \to \mathbb{R}^{n_y} \tag{2}$$

2.2 Local Approximations and Points of Expansion

The starting point and point of expansion for local approximations to the solution of DSGE models is the deterministic steady state, defined as follows

Definition 2.1. *Deterministic Steady State*

Let $\bar{y}^{det} \in \mathbb{R}^{n_y}$ *define a fixed point of (2) given by*

$$\bar{y}^{det} = g(\bar{y}^{det}, 0, 0) \tag{3}$$

i.e., a fixed point of (2) in the absence of both risk $(\sigma = 0)$ *and shocks* $(\varepsilon_t = 0)$*.*

The deterministic steady state $\bar{y}$ is recovered by solving a deterministic version of (1) in the absence of risk and shocks $0 = f(\bar{y}, \bar{y}, \bar{y}, 0)$.

DSGE perturbation then constructs a Taylor series expansion of the policy function, (2), up to some, say, M'th order around the deterministic steady state, given by[10]

$$y_t \approx \sum_{j=0}^{M} \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^j\sigma^i} \sigma^i \right] (z_t - \bar{z})^{\otimes[j]} \tag{4}$$

where $g_{z^j\sigma^i} \doteq \mathscr{D}^{j+i}_{z^j_{t-1}\sigma^i}\{y(\sigma, z_t)\} \in \mathbb{R}^{n_y \times n_z^j}$, with $n_z = n_y + n_\varepsilon$ is the partial derivative of the vector function y with respect to the state vector $z_t \doteq \begin{bmatrix} y'_{t-1} & \sigma\varepsilon'_t \end{bmatrix}'$ j times and the perturbation parameter σ i times[11] evaluated at the deterministic steady state using the notation outlined in appendix A.2.

cesses can be captured by including the processes themselves in the vector y_t and including functions in f that specify the nonlinearity or correlation pattern.

[9] My formulation follows Adjemian, Bastani, Juillard, Mihoubi, Perendia, Pfeifer, Ratto, and Villemot's (2011) Dynare, Anderson, Levin, and Swanson's (2006) PerturbationAIM and Juillard (2011). Jin and Judd's (2002) or Schmitt-Grohé and Uribe's (2004) model classes can be rearranged to fit (1). Furthermore, my scaling of all ε_t, future, past, and present, follows Lombardo (2010).

[10] The assumptions that validate this expansion will be introduced in section 2.3. See Lan and Meyer-Gohde (Forthcoming) for a derivation of this multivariate Taylor series approximation.

[11] As the perturbation parameter also scales ε_t, I should say this is the partial derivative with respect to the third

The deterministic steady state is the fixed point for the deterministic, $\sigma = 0$, model but not the stochastic, $\sigma = 1$, model. This manifests itself in the Taylor series (4) as the constant terms $\sum_{i=1}^{M} \frac{1}{i!} g_{\sigma^i} \sigma^i$ that move the Taylor series away from the deterministic steady state. The deterministic steady state is neither a fixed point of the stochastic model and nor of the Taylor series approximation of the stochastic, $\sigma = 1$, model for $M \geq 2$, when σ corrects for the second moment of risk.

The stochastic steady state is the steady state of the stochastic model, incorporating risk into its definition in the following

Definition 2.2. *Stochastic Steady State*

Let $\overline{y}^{stoch} \in \mathbb{R}^{n_y}$ *define a fixed point of (2) given by*

$$\overline{y}^{stoch} = g(\overline{y}^{stoch}, 0, 1) \tag{5}$$

i.e., a fixed point of (2) in the presence of risk $(\sigma = 1)$ *but in the absence of shocks* $(\varepsilon_t = 0)$*.*

That is, the fixed point in the state space in the absence of shocks, but while expecting future shock with a known probability distribution. Solving for an approximation of the stochastic steady state is not trivial using the state space formulation of the policy function, see Coeurdacier, Rey, and Winant (2011), Juillard (2011), and de Groot (2013), and the definition in (5) is not directly useful. The difficulty arises as $\overline{y}^{stoch}$ is defined only implicitly in (5) and the method of solving $0 = f(\overline{y}, \overline{y}, \overline{y}, 0)$ to recover the deterministic steady state is not available as the presence of risk requires the the integral over the probability distribution of future shocks embodied by the expectations operator be maintained in (1). Alternatively, one could examine an alternative nonlinear moving average representation of the policy function that results upon inverting or recursively substitution the state space policy function (2).[12] Approximated out to some, say, M'th order around

argument and write $g_{[1\ 2]'^j 3^i} \in \mathbb{R}^{n_y \times n_z^j}$, but I choose not to do so as not to overload the notation. The complete, direct and indirect, derivative of y_t with respect to σ, $\mathscr{D}_\sigma\{y_t\}$, in my notation is given by $\mathscr{D}_\sigma\{y_t\} = y_\varepsilon \varepsilon_t + y_\sigma$

[12]See Lan and Meyer-Gohde (2013c) for solving and analyzing DSGE models with nonlinear moving averages and Aruoba, Bocola, and Schorfheide (2013) for an empirical application of Volterra series to macroeconomic time series. Again the the assumptions that validate this expansion will be introduced in section 2.3.

the deterministic steady state as[13]

$$(6) \qquad y_t \approx \sum_{m=0}^{M} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 i_2 \cdots i_m} \sigma^n \right] (\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \cdots \otimes \varepsilon_{t-i_m})$$

where $y_{\sigma^n i_1 \cdots i_m} \sigma^n$ is the derivative of y_t with respect to the m'th fold Kronecker products of exogenous innovations i_1, i_2, ... and i_m periods ago and with respect to the perturbation parameter, σ, n times. The stochastic steady state now follows by letting the history of shocks be equal to zero at all dates (i.e., letting y_t converge to its fixed point), but letting $\sigma = 1$ to correct for risk to M'th order, yielding $\overline{y}^{stoch} \approx \sum_{n=0}^{M} \frac{1}{n!} y_{\sigma^n}$.

Although not a fixed point in nonlinear stochastic settings, the ergodic mean of y_t is a potentially empirically relevant point around which one might consider constructing a local approximation, defined as

Definition 2.3. *Ergodic Mean*
Let $\overline{y}^{mean} \in \mathbb{R}^{n_y}$ *be a vector such that*

$$(7) \qquad \overline{y}^{mean} \doteq E\left[y_t\right] = E\left[g(y_{t-1}, \varepsilon_t, 1)\right]$$

being the unconditional expectation of (2) in the presence of uncertainty $(\sigma = 1)$ *and shocks* (ε_t)*.*

This point would be particulary advantageous in empirical applications as it is likely to be associated with a high probability density. Again, the definition in (7) is not particularly useful, as calculating the mean requires integration over the endogenous variables through the unknown policy function. The nonlinear moving average formulation proves advantageous again, requiring only knowledge of the moments of exogenous variables to give an approximation of the ergodic mean

$$(8) \qquad \overline{y}^{mean} \doteq E\left[y_t\right] \approx \sum_{m=0}^{M} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 i_2 \cdots i_m} \sigma^n \right] E\left[\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \cdots \otimes \varepsilon_{t-i_m}\right]$$

to calculate the mean.

In section 3, I will calculate risky linear approximations around the stochastic steady state and the ergodic mean, approximating these points and the derivatives of the policy function at these points through implicit differentiation of the policy function (2) at the deterministic steady state.

[13] See Lan and Meyer-Gohde (2013b) for the mapping between the partial derivatives $g_{z^j \sigma^i}$ and $y_{\sigma^n i_1 i_2 \cdots i_m}$.

That is, I will derive the points and slopes for the risky linear approximation jointly using a new method and I will compare the results with those presented here. Before I derive these risky linear approximations, I will layout the assumptions necessary to validate the procedure.

2.3 Assumptions and Validity of Risky Linear Approximation

I will present assumptions for the local approximation of the model in (1) around the deterministic steady state and for this approximation to be valid at the stochastic steady state and ergodic mean. Then I will show how these assumptions validate the methodology that I will use in the next section of constructing a local approximation of the stochastic or "risky" model around the stochastic steady state or ergodic mean by using derivative information from the deterministic model evaluated at the deterministic steady state.

I summarize my assumptions in the following

Assumption 2.4.

1. *Differentiability: the functions f in (1) and g in (2) are C^M with respect to all their arguments, where M is the order of approximation to be introduced subsequently*

2. *Exogenous Moments: the elements of ε_t are i.i.d. with $E[\varepsilon_t] = 0$ and $E\left[\varepsilon_t^{\otimes[m]}\right]$ finite $\forall m \leq M$, where $\varepsilon_t^{\otimes[m]}$ is the m'th fold Kronecker product of ε_t with itself: $\underbrace{\varepsilon_t \otimes \varepsilon_t \cdots \otimes \varepsilon_t}_{m \ times}$*

3. *Local Analyticity: the function g is locally analytic around the deterministic steady state ($y_{t-1} = \bar{y}$, $\varepsilon_t = 0$, $\sigma = 0$) with a domain of convergence that contains the stochastic steady state and ergodic mean*

4. *Local Stability: the eigenvalues of g_y evaluated at the deterministic steady state ($y_{t-1} = \bar{y}$, $\varepsilon_t = 0$, $\sigma = 0$) are all inside the unit circle.*

The first assumption ensures that the functions involved are smooth out to the order of approximation, the second that the exogenous process is defined at least out to the order of approximation,[14]

[14] Jin and Judd (2002) would additionally require bounded support for the exogenous process. Kim, Kim, Schaumburg, and Sims (2008) offer skepticism regarding the necessity of a boundedness assumption.

the third that the true policy function has an infinite order Taylor series representation that remains valid as the state space moves to the stochastic steady state and ergodic mean, and that the solution is locally stable at the deterministic steady state.

From the assumption of local analyticity, y_t has the Taylor series representation[15]

$$y_t = \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{i=0}^{\infty} \frac{1}{i!} g_{z^j \sigma^i} \sigma^i \right] \left[y'_{t-1} - \bar{y} \quad \sigma \varepsilon'_t \right]^{\prime \otimes [j]} \tag{9}$$

From the assumption of local analyticity as well as of local stability, it follows that it admits an infinite nonlinear moving average or Volterra series representation[16]

$$y_t = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{n!} y_{\sigma^n i_1 \cdots i_m} \sigma^n \right] (\sigma \varepsilon_{t-i_1} \otimes \cdots \otimes \sigma \varepsilon_{t-i_m}) \tag{10}$$

Local analyticity guarantees that increasing the approximation order in the standard perturbation brings the approximation closer to the true policy function in the sense that an infinite order perturbation will recover the true policy function.

The assumption that analyticity holds over a domain larger than the deterministic steady state is necessary, as standard perturbation starting with the second order entail risk corrections to the constant term, see Schmitt-Grohé and Uribe (2004) among others, which moves the fixed point of the model away from the deterministic steady state.[17]. Additionally, if the model is simulated, the model will tend towards the ergodic mean and will vary with shock realizations in that vicinity. Hence, for standard perturbations to be applicable in settings useful for analysis, they must maintain their validity in a region that contains the deterministic steady state, stochastic steady state, and ergodic mean.

In this case, both the stochastic steady state

$$\bar{y}^{\text{stoch}} = \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{i=0}^{\infty} \frac{1}{i!} g_{z^j \sigma^i} \right] \left[\bar{y}^{\text{stoch}\prime} - \bar{y} \quad 0' \right]^{\prime \otimes [j]} \tag{11}$$

[15] Additionally, Lan and Meyer-Gohde (Forthcoming) prove that assumptions 2.4 are sufficient to guarantee the solvability of DSGE perturbations; that is, that standard DSGE perturbation practice of successively differentiating the equilibrium conditions in (1) will deliver equations that can be uniquely solved for the coefficients, $g_{z^j \sigma^i}$, in (9).

[16] See Lan and Meyer-Gohde (2013c) and Sandberg (1983).

[17] For pruned perturbations, see Kim, Kim, Schaumburg, and Sims (2008) and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013), Lan and Meyer-Gohde (2013b) the new fixed point is the stochastic steady state

and the ergodic mean

(12) $$E\left[y_t\right] = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{1}{n!}y_{\sigma^n i_1\cdots i_m}\right]E\left[\varepsilon_{t-i_1}\otimes\cdots\otimes\varepsilon_{t-i_m}\right]$$

are recoverable from the implicit function theorem, requiring only derivatives of g evaluated at the deterministic steady state as enter the Taylor series (9) and the moments of ε_t. Furthermore, due to the analyticity assumed in a domain containing both the ergodic mean and the stochastic steady state, the policy function is analytic at these points as well, ensuring its infinite differentiability there as well. If the function is infinitely differentiable, its first derivatives must also exist there as well, hence validating the the first derivatives used in the risky linear approximation.

Thus, under the assumption that standard perturbations with ever increasing order will recover the true policy function in a domain that contains the point of approximation, the deterministic steady state, and two points in the state space useful for analysis, the stochastic steady state and the ergodic mean, these two points and the first derivative of the policy function at these points can be recovered using derivative information at the deterministic steady state and the moments of the exogenous shocks of the model. Now I shall proceed to do exactly that and assemble the points and slopes into linear approximations.

3 Risky Linear Approximations

Now I will define a risky linear approximation, that is, an approximation that is linear in the states, y_{t-1} and ε_t but corrected to arbitrary order for risk, i.e., out to the desired moments of the distribution of the exogenous shocks ε_t in (1). The correction for risk is accomplished by expanding the policy function nonlinearly in σ, the index that scales risk. Expanding to first order in σ corrects for the first moment of ε_t,[18] to second order in σ corrects for the second moment of ε_t, and so forth.

I will write such a σ dependent or risky linear approximation as

(13) $$y_t \simeq \overline{y}(\sigma) + y_y(\sigma)\left(y_{t-1} - \overline{y}(\sigma)\right) + y_\varepsilon(\sigma)\varepsilon_t$$

[18] As the shock is assumed mean zero, this correction does not alter the policy functions compared with their deterministic counterparts. See Schmitt-Grohé and Uribe (2004) and Lan and Meyer-Gohde (Forthcoming).

where $\overline{y}(\sigma)$ is a σ dependent or risky point for y_t and $y_y(\sigma)$ and $y_\varepsilon(\sigma)$ are the σ dependent or risky first derivatives of y_t at this σ dependent or risk-corrected point. I shall consider two risky points, the stochastic steady state and the ergodic mean.

From standard DSGE perturbation, I have derivative information at the deterministic steady state. I will now show that the first derivatives, $y_y(\sigma)$ and $y_\varepsilon(\sigma)$, will depend on the σ dependent point of interest $\overline{y}(\sigma)$ and that two natural choices for this point of interest, the stochastic steady state and the ergodic mean, along with these first derivatives can be obtained from derivative information obtained at the deterministic steady state.

3.1 Risky Points of Approxiamtion

3.1.1 Stochastic Steady State

The first point around which I will construct a linear approximation is the stochastic steady state.

The stochastic and the deterministic steady states can be embedded implicitly in a σ-dependent steady state

$$\overline{y}(\sigma) = g(\overline{y}(\sigma), 0, \sigma) \tag{14}$$

Here, $\sigma = 1$ gives the stochastic and $\sigma = 0$ the deterministic steady state. A Taylor expansion of $\overline{y}(\sigma)$ around the $\sigma = 0$ deterministic steady state can be written as

$$\overline{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \overline{y}_{\sigma^i}(0)\, \sigma^i \tag{15}$$

Evaluated at $\sigma = 1$ to recover the stochastic steady state and discarding terms of order higher than two to focus on the effects of the first two moments of the stochastic process gives

$$\overline{y}(\sigma) = \overline{y} + \overline{y}_\sigma(0) + \frac{1}{2}\overline{y}_{\sigma^2}(0) + O(\sigma^3) \tag{16}$$

Using the Taylor expansion in σ, the stochastic steady state can be approximated by solving sets of linear equations with inhomogeneous constants collecting lower order terms and standard DSGE perturbation output, as I summarize in the following

Proposition 3.1. *σ Approximation of the Stochastic Steady State*

Let assumption 2.4 hold, the stochastic steady state in (5) as represented by (15) can be approximated in σ using only derivative information from standard perturbations—$g_{z^j\sigma^i}$ in (4)—around the deterministic steady state.

Proof. See appendix A.3. □

To capture the effect of the first two moments of the exogenous processes (i.e., that of the variance of the mean zero growth shocks in the model of section 4) on the stochastic steady state, a second order approximation in σ is needed, given in closed form by the following

Corollary 3.2. *Second Order σ Approximation of the Stochastic Steady State*
The stochastic steady state in (5) can be approximated to second order, in σ, as

$$\overline{y}^{stoch} \approx \overline{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1} g_{\sigma^2} \quad (17)$$

Proof. See appendix A.4. □

The value reported in (17) is identical to the value reported in Lan and Meyer-Gohde (2013b) derived using a second-order nonlinear moving average.

3.1.2 Ergodic Mean

The second point around which I would like to be able to construct a linear approximation is the ergodic mean.

The ergodic mean and the deterministic steady state can be embedded implicitly in a σ-dependent point

$$\overline{y}(\sigma) \doteq E\left[g(y_{t-1}, \sigma\varepsilon_t, \sigma)\right] \quad (18)$$

Here, $\sigma = 1$ gives the ergodic mean and $\sigma = 0$ the deterministic steady state. Due to the singularity induced by $\sigma = 0$, which turns off the stochastics in the model, the steady state coincides with the mean in this deterministic setting, which I exploit to extrapolate from the deterministic steady state to the ergodic mean. A Taylor expansion of $\overline{y}(\sigma)$ around the $\sigma = 0$ deterministic steady state can be written as

$$\overline{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!}\overline{y}_{\sigma^i}(0)\,\sigma^i \tag{19}$$

Evaluated at $\sigma = 1$ to recover the ergodic mean and discarding terms of order higher than two to focus on the effects of the first two moments of the stochastic process gives

$$\overline{y}(\sigma) = \overline{y} + \overline{y}_{\sigma}(0) + \frac{1}{2}\overline{y}_{\sigma^2}(0) + O(\sigma^3) \tag{20}$$

Using the Taylor expansion in σ, the ergodic can be approximated by solving sets of linear equations with inhomogeneous constants collecting lower order terms, standard DSGE perturbation output, and the given moments of the exogenous driving force, as I summarize in the following

Proposition 3.3. *σ Approximation of the Ergodic Mean*

Let assumption 2.4 hold, the ergodic mean in (7) as represented by (19) can be approximated in σ using derivative information from standard perturbations—$g_{z^j\sigma^i}$ in (4)—evaluated at the deterministic steady state and the given finite moments of the exogenous driving force.

Proof. See appendix A.5. □

To capture the effects of the first two moments of the exogenous processes (i.e., that of the variance of the mean zero growth shocks in the model of section 4) on the stochastic steady state, a second order approximation in σ is needed, given in closed form by the following

Corollary 3.4. *Second Order σ Approximation of the Ergodic Mean*

The ergodic mean in (7) can be approximated to second order, in σ, as

$$\overline{y}^{mean} \approx \overline{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_{\varepsilon}^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right) \tag{21}$$

Proof. See appendix A.6. □

The value reported in (21) is identical to the value reported in Lan and Meyer-Gohde (2013a) using a second nonlinear moving average and in Andreasen, Fernández-Villaverde, and Rubio-Ramírez

(2013) using a second order pruned state space approximation.[19]

3.2 Risky First Derivatives

Given a risky point from above, either the stochastic steady state from proposition 3.1 or the ergodic mean from proposition 3.3, I need only to calculate the first derivatives with respect to states and shocks in order to complete the construction of the risky linear approximation in (13).

The first derivatives with respect to states and shocks around a risky point, $\overline{y}(\sigma)$ from above, are given by

Definition 3.5. *First Derivatives at a σ Adjusted Point*

The derivatives of y_t with respect to y_{t-1} and ε_t at a risky point, $\overline{y}(\sigma)$ are

$$\overline{y_y}(\sigma) \doteq g_y(\overline{y}(\sigma), 0, \sigma), \quad \overline{y_\varepsilon}(\sigma) \doteq g_\varepsilon(\overline{y}(\sigma), 0, \sigma) \tag{22}$$

The first derivatives are σ dependent functions, both directly and indirectly through the risky point of approximation, $\overline{y}(\sigma)$, chosen. Here, $\sigma = 1$ gives the first derivatives at the risky point of approximation and $\sigma = 0$ at the deterministic steady state. Taylor expansions of $\overline{y_y}(\sigma)$ and $\overline{y_\varepsilon}(\sigma)$ around the $\sigma = 0$ deterministic steady state can be written as

$$\overline{y_y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!}\overline{y}_{y\sigma^i}(0)\,\sigma^i, \quad \overline{y_\varepsilon}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!}\overline{y}_{\varepsilon\sigma^i}(0)\,\sigma^i \tag{23}$$

Evaluated at $\sigma = 1$ to recover the derivatives at the risky point of approximation and discarding terms of order higher than two to focus on the effects of the first two moments of the stochastic process gives

$$\overline{y_y}(\sigma) = \overline{y_y} + \overline{y}_{y\sigma}(0) + \frac{1}{2}\overline{y}_{y\sigma^2}(0) + O(\sigma^3), \quad \overline{y_\varepsilon}(\sigma) = \overline{y_\varepsilon} + \overline{y}_{\varepsilon\sigma}(0) + \frac{1}{2}\overline{y}_{\varepsilon\sigma^2}(0) + O(\sigma^3) \tag{24}$$

As was the case with the two risky points considered above, the first derivatives at these points depend only on standard output from perturbation algorithms: derivatives of the policy function at the deterministic steady state and the moments (through the derivatives of the risky ergodic mean)

[19] See also Kim, Kim, Schaumburg, and Sims (2008) for second order pruning and Lombardo (2010) for a theoretical justification of state space pruning. Lan and Meyer-Gohde (2013b) provide an overview and comparison of pruning in the literature.

of the exogenous shocks, ε_t, as I summarize in the following

Proposition 3.6. *σ Approximation of the First Derivatives*

The first derivatives in (22) can be approximated in σ using only derivative information from standard perturbations—$g_{z^j\sigma^i}$ in (4)—evaluated at the deterministic steady state and the derivative information in σ from the chosen risky point of approximation.

Proof. Successive differentiation of (22) yields $\overline{y}_{y\sigma^i}$ and $\overline{y}_{\varepsilon\sigma^i}$ as functions of derivatives of g with respect to y_{t-1} and σ as well as derivatives of the chosen risky point of approximation $y(\sigma)$. □

Approximating out to second order in σ as above for the risky points of approximation to adjust the first derivatives for the first two moments of the the distribution of the exogenous shocks, ε_t, gives the following

Corollary 3.7. *Second Order σ Approximation of the First Derivatives*

The first derivatives in (22) can be approximated to second order, in σ, as

$$\overline{y_y}(1) \approx g_y + \frac{1}{2}\left(g_{y^2}\left(\overline{y}''(0) \otimes I_{n_y}\right) + g_{\sigma^2 y}\right), \quad \overline{y_\varepsilon}(1) \approx g_\varepsilon + \frac{1}{2}\left(g_{y\varepsilon}\left(\overline{y}''(0) \otimes I_{n_e}\right) + g_{\sigma^2\varepsilon}\right) \tag{25}$$

Proof. See appendix A.7. □

With the risky first derivatives in hand, the risky linear approximation in (13) can be constructed by choosing either the approximation of the stochastic steady state or of the ergodic mean and calculating the associated first derivatives. Before I turn to an application of the approximation in an economic model, I will address the theoretical validity of the approximation.

3.2.1 Relation to Perturbation First Derivatives

Here, I examine the relationship of the risky points and slopes derived above with standard perturbation output. I show that the risky first derivatives correctly recover the first derivatives of the Taylor series evaluated at the risky points above out to a given order in σ.[20]

[20]When the risky point is the stochastic steady state, the first derivatives recover the risky derivatives used in Lan and Meyer-Gohde's (2013b) recursive formulation of Lan and Meyer-Gohde's (2013c) nonlinear moving average. These

Differentiate the Taylor series representation of the standard perturbation in (4) with respect to the state vector z_t to yield

$$\mathscr{D}_{z_t} y_t \approx \sum_{j=1}^{M} \frac{1}{(j-1)!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^j \sigma^i} \sigma^i \right] \left[(z_t - \overline{z})^{\otimes [j-1]} \otimes I_{n_z} \right]$$

$$\approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1} \sigma^i} \sigma^i \right] \left[(z_t - \overline{z})^{\otimes [j]} \otimes I_{n_z} \right] \tag{26}$$

Evaluated at the deterministic steady state, $z_t = \overline{z}$, the foregoing collapses to

$$\mathscr{D}_{z_t} y_t \approx \sum_{i=0}^{M-1} \frac{1}{i!} g_{z \sigma^i} \sigma^i \tag{27}$$

For $M = 3$, the first derivative from a third order perturbation approximation, setting σ to one and recalling that terms first order in σ are zero,[21] is

$$\mathscr{D}_{z_t} y_t \approx g_z + \frac{1}{2} g_{z\sigma^2} \tag{28}$$

The last term, $g_{z\sigma^i}$, is the time varying risk correction that enters at third order in the pruning algorithm of Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013) and matched perturbation of Lombardo (2010)—see Lan and Meyer-Gohde (2013b) for a detailed comparison of these and other pruning algorithms.

Of primary interest here are the derivatives at the risky points of interest derived in section 2. Recall, (15) and (19), that the risky points can be expressed as Taylor series in σ; i.e.,

$$\overline{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \overline{y}_{\sigma^i}(0) \sigma^i \tag{29}$$

setting $y_t = \overline{y}(\sigma)$ and $\varepsilon_t = 0$, in the first derivative of the Taylor series representation of the policy function from (26) yields

$$\mathscr{D}_{z_t} y_t \approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1} \sigma^i} \sigma^i \right] \left(\begin{bmatrix} \overline{y}(\sigma) - \overline{y} \\ 0 \end{bmatrix}^{\otimes [j]} \otimes I_{n_z} \right) \tag{30}$$

For a second order in σ approximations of the points of interest derived explicitly in section 2, this

derivatives differ from those used in Andreasen, Fernández-Villaverde, and Rubio-Ramírez's (2013) pruning algorithm, which give risky derivatives evaluated at the deterministic steady state.

[21] See Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and Lan and Meyer-Gohde (Forthcoming).

expression becomes

$$\begin{aligned}\mathscr{D}_{z_t} y_t &\approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1}\sigma^i} \sigma^i \right] \left(\begin{bmatrix} \frac{1}{2}\overline{y}_{\sigma^2}(0)\,\sigma^2 \\ 0 \end{bmatrix}^{\otimes[j]} \otimes I_{n_z} \right) \\ &\approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1}\sigma^i} \right] \left(\begin{bmatrix} \frac{1}{2}\overline{y}_{\sigma^2}(0) \\ 0 \end{bmatrix}^{\otimes[j]} \otimes I_{n_z} \right) \sigma^{i+2j}\end{aligned} \tag{31}$$

Discarding terms in σ of order higher than two in order to obtain a second order in σ approximation of the matrix of first derivatives at the risky point of interest gives

$$\mathscr{D}_{z_t} y_t \approx g_z + \frac{1}{2}\left[g_{z\sigma^2} + g_{z^2}\left(\begin{bmatrix} \overline{y}_{\sigma^2} \\ 0 \end{bmatrix} \otimes I_{n_z} \right) \right] \tag{32}$$

or in terms of derivatives with respect to y_{t-1} and ε_t separately

$$\mathscr{D}_{y_{t-1}} y_t \approx g_y + \frac{1}{2}\left[g_{y\sigma^2} + g_{y^2}\left(\overline{y}_{\sigma^2} \otimes I_{n_y} \right) \right] \tag{33}$$

$$\mathscr{D}_{\varepsilon_t} y_t \approx g_\varepsilon + \frac{1}{2}\left[g_{\varepsilon\sigma^2} + g_{y\varepsilon}\left(\overline{y}_{\sigma^2} \otimes I_{n_e} \right) \right] \tag{34}$$

which are identical to the results presented in section 3.2. The risky derivatives derived in section 3.2 with my implicit risk adjustment procedure provide the correct derivatives at the chosen risky point of interest up to the chosen order in σ.

4 Long Run Risk and the Real Business Cycle

I will examine a canonical real business cycle model in the spirit of Kydland and Prescott (1982) with two differentiating features to emphasize the role of risk: recursive—or risk-sensitive—preferences and long run risk. To introduce risk sensitivity, I will follow Epstein and Zin (1989), Weil (1990), and others by replacing the continuation value of household utility with a power certainty equivalent, which allows me to disentangle risk aversion and the inverse elasticity of intertemporal substitution. I confront agents with long run real risk in the form of stochastic trends in productivity,[22] which adds risk to the growth of consumption, making the stochastic driving force of model welfare relevant in the sense of Lucas (1987). My choice of model is very similar to the model used in the numerical study of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), where I have replaced

[22] See, e.g., Bansal and Yaron (2004) in an endowment and Rudebusch and Swanson (2012) in a production model.

their stochastic volatility with long run risk.[23]

The social planner maximizes the expected discounted lifetime utility of a representative household given recursively[24] by

$$U_t = \max_{C_t, L_t} \left[(1-\beta) \left(C_t^{\nu} (1-L_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}} \tag{35}$$

where C_t is consumption, L_t labor, $\beta \in (0,1)$ the discount factor, ν a labor supply parameter, γ governs risk aversion,[25] and

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} \tag{36}$$

where ψ is the elasticity of intertemporal substitution (IES). Thus, these recursive preferences disentangle the IES and risk aversion. The social planner faces the resource constraint

$$C_t + K_t = K_{t-1}^{\xi} \left(e^{Z_t} L_t \right)^{1-\xi} + (1-\delta) K_{t-1} \tag{37}$$

with K_t being capital, ξ its output elasticity and δ its depreciation rate, and Z_t a random walk with drift productivity process given by

$$a_t \doteq Z_t - Z_{t-1} = \overline{a} + \overline{\sigma}\varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0,1) \tag{38}$$

with $\overline{\sigma}$ the standard deviation of a_t and $\overline{a}$ the drift.

The first order conditions are the intratemporal condition

$$\frac{1-\nu}{\nu} \frac{C_t}{1-L_t} = (1-\xi) e^{(1-\xi)Z_t} K_{t-1}^{\xi} L_t^{-\xi} \tag{39}$$

and the intertemporal condition

$$1 = E_t \left[M_{t+1} \left(\xi K_t^{\xi-1} \left(e^{Z_{t+1}} L_{t+1} \right)^{1-\xi} + 1 - \delta \right) \right] \tag{40}$$

[23]As I will be examining linear approximations, I would only be able to capture the precautionary effects or average effects of stochastic volatility and would miss the time varying effects of changes in conditional heteroskedasticity. One could conceivably move the approximation towards the conditionally normal one used in Justiniano and Primiceri (2008), to retain some of the advantages of linearity, but this is beyond the scope of this paper.

[24]See Epstein and Zin (1989) and Weil (1990). Backus, Routledge, and Zin (2005) provide a detailed overview of these and other preferences that deviate from standard expected utility.

[25]In the presence on an adjustable labor margin, the standard measure of risk aversion does not directly apply, see Swanson (2012a). Swanson (2012b) presents measures of risk aversion under recursive preferences in the presence of a labor margin. I maintain the misnomer of referring to γ as risk aversion for expositional ease.

where the pricing kernel is given by

$$M_{t+1} \doteq \frac{\partial U_t/\partial C_{t+1}}{\partial U_t/\partial C_t} = \beta \frac{C_t}{C_{t+1}} \frac{\left(C_{t+1}^{\nu}\left(1-L_{t+1}\right)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}}}{\left(C_t^{\nu}\left(1-L_t\right)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}}} \left(\frac{U_{t+1}^{1-\gamma}}{E_t\left[U_{t+1}\right]^{1-\gamma}}\right)^{1-\frac{1}{\theta}} \tag{41}$$

The presence of U_{t+1} in the pricing kernel necessitates the inclusion of the value function evaluated at the optimum, where I recycle the notation U_t,

$$U_t = \left[(1-\beta)\left(C_t^{\nu}\left(1-L_t\right)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}} + \beta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\theta}{1-\gamma}} \tag{42}$$

along with the first order conditions, the resource constraint (37), and the exogenous driving force (38) to characterize an equilibrium. With the stochastic trend in the model, I detrend all variables (apart from L_t and M_t) with $x_t \doteq X_t/e^{Z_t}$; the value function is detrended slightly differently, $u_t \doteq U_t/e^{\nu Z_t}$.[26] I reexpress all macroeconomic variables through a log transformation $\hat{x}_t = \ln(x_t)$ so that deviations in these variables from any given value can be interpreted as percentage deviations and a linear approximation of $\hat{x}_t$ gives a log linear approximation.[27]

Additionally, I will examine two conditional asset pricing variables to measure shifts in the pricing of risk. First, the ex ante risk premium

$$erp_t = E_t\left[r_{t+1}^k - r_t^f\right] \tag{43}$$

where the risk-free rate, r_t^f, is given by $r_t^f \doteq \frac{1}{E_t[M_{t+1}]}$ and the return on capital, r_t^k, is given by $r_t^f \doteq \xi K_{t-1}^{\xi-1}\left(e^{Z_t}L_t\right)^{1-\xi} + 1 - \delta$; and second, the (squared) conditional market price of risk

$$cmpr_t = \frac{E_t\left[\left(M_{t+1} - E_t\left[M_{t+1}\right]\right)^2\right]}{E_t\left[M_{t+1}\right]^2} \tag{44}$$

I will take the square of the usual conditional market price of risk—conditional standard deviation over conditional mean—to maintain differentiability at the deterministic steady state, necessary for the application of the perturbation methods to which I will turn to later. Finally, I will include the ex post risk premium

$$rp_t = r_t^k - r_{t-1}^f \tag{45}$$

as an observable counterpart to the ex ante version above.

[26] See appendix A.1 for the detrended model.
[27] See, e.g., Uhlig (1999).

5 Assessing the Accuracy of the Risky Linear Approximation in a Calibrated Model

I now apply the risky linear approximation to two calibrated versions of the neoclassical growth model of section 4. I begin by assessing the accuracy of the risky linear approximation by examining the Euler equation errors and comparing the results to standard linear and higher order perturbation solutions.

5.1 Calibration

In table 1, I report the parameter values common to both calibrations I consider here. The calibration for these parameters largely follows Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) and reflects standard observations on the post war US economy: ξ is set to match the labor share of national income; β to reflect an annual interest rate of about 3.5 %; the value of ν induces work to occupy roughly one third of the time endowment; and δ aligns the model in steady state to the investment output ratio. The value of $\overline{a}$ is set to match the average postwar growth rate of output.

[Table 1 about here.]

In table 2, I report the baseline calibration in the first three columns. Here, I set risk aversion, γ, to 5, following the baseline parameterization of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012). The elasticity of intertemporal substitution (IES) and the standard deviation of technology growth shocks are set to match the standard deviations of log consumption and output growth for the third order in perturbation solution of the model. This value for the IES lies in the range of 0.5 to 1.5 examined in Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), reflecting conservative bounds on the parameter advocated in the literature.

[Table 2 about here.]

The extreme calibration can be found in the last three columns of table 2. Risk aversion, γ, is equal to 40, following Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012). The IES and the standard deviation of technology growth shocks are again set to match the standard deviations of log consumption and output growth for the third order in perturbation solution of the model. When calibrating to the two macro data sets, the values for the IES and the standard deviation of technology growth shocks remain virtually unchanged, reflecting the well known result—see, e.g., Tallarini (2000)—that macro series are driven primarily by the IES and not risk aversion.

5.2 Euler Equation Errors

Following Judd and Guu (1997) and Judd (1998), I use the unit-free Euler equation residuals as a measure of accuracy.[28] I will assess the accuracy of the risky linear approximation at the approximated stochastic steady state.[29]

The Euler equation error expressed as a fraction of time t consumption is given by

(46)

$$EEE(k_{t-1},\varepsilon_t) = \frac{E_t\left[\beta\left(c_{t+1}e^{a_{t+1}}\right)^{\nu\frac{1-\gamma}{\theta}-1}\left(\frac{1-L_{t+1}}{1-L_t}\right)^{\frac{(1-\nu)(1-\gamma)}{\theta}}\left(\frac{\left(v_{t+1}e^{\nu a_{t+1}}\right)^{1-\gamma}}{E_t\left[\left(v_{t+1}e^{\nu a_{t+1}}\right)^{1-\gamma}\right]}\right)^{1-\frac{1}{\theta}}\left(\alpha k_t^{\alpha-1}\left(e^{a_{t+1}}L_{t+1}\right)^{1-\alpha}+1-\delta\right)\right]^{\frac{1}{\nu\frac{1-\gamma}{\theta}-1}}}{c_t} - 1$$

Here, the value of, say, $1E-2$ implies a \$1 mistake for each \$100 spent and the value of $1E-3$ implies a \$1 mistake for each \$1000 and so forth. This is a function of the state, k_{t-1} and shock, ε_t. In figure 1, I plot the Euler equation errors with the current shock set to zero and examine how this error depends on the endogenous state, k_{t-1}.

[Figure 1 about here.]

In figure 1a, the Euler equation errors for the baseline calibration can be found. The risky linear approximation uniformly improves on the linear approximation, while lagging behind the second and third order perturbation. In general, higher order perturbations improve the accuracy of the approximation. In the vicinity of the steady states,[30] this improvement is more than one order of

[28] See also Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) and Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) for applications of this measure to assess the accuracy of varying solution methods.

[29] The results are virtually unchanged with the alternative, the ergodic mean.

[30] Under the baseline calibration, the stochastic and determinisitic steady states are nearly the same.

magnitude. In sum, the risky linear approximation is, while still linear in states and shocks, is uniformly more accurate than the standard linear approximation.

The Euler equation errors for the extreme calibration are depicted in figure 1b. As the importance of risk in the underlying nonlinear system is increased through the increase in risk aversion in the extreme calibration, so does the relative performance of the risky linear approximation. Now the risky linear approximation is two orders of magnitude more accurate than the standard linear approximation over a broad vicinity surrounding the steady states.[31] Furthermore, the risky linear approximation is roughly comparable to higher order approximations despite its linearity in states and shocks.

Thus, for small shock realizations and values of the state, capital, close to the stochastic steady state, I conclude that the risky linear approximation of the previous sections outperforms the standard linear approximation and performs favorably compared with higher order perturbations.

5.3 Impulse Response Analysis

Having established the satisfactory accuracy of the risky linear approximation, I will turn to its structural predictions in the form of impulse response functions. A significant advantage of the linearity in states and shocks of the approximation is that an impulse response is a straightforward concept. Whereas nonlinear methods must take a stance regarding the specific assumptions regarding a generalized impulse response,[32] the risky linear approximation and its standard linear approximation require no such discussion.

[Figure 2 about here.]

In figure 2, the impulses of selected macroeconomic and financial variables with respect to a one standard deviation shock to the growth rate of technology are plotted. Figure 2a contains the

[31]Now the difference between the stochastic and deterministic steady states can be discerned visually, with the minimum of the risky linear approximation, the stochastic steady state, to the right of the minimum of the standard linear approximation, the deterministic steady state.

[32]See Lan and Meyer-Gohde (2013c) and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013) and the specific assumptions therein.

impulses of consumption and capital to a growth rate shock. Impulse responses from the risky linear (here the stochastic steady state version), standard linear, and third order nonlinear moving average approximations are indistinguishable up to numerical rounding. Consumption and capital, both detrended, fall in response to the shock as the capital stock and consumption catch up to the accelerated growth path. Figure 2b contains the impulses of the expected risk premium and the conditional market price of risk. The standard linear approximation fails to capture the movement in these conditional asset pricing variables, while the risky linear approximation put forth in previous sections here matches, up to numerical rounding, the impulses generated by the full nonlinear third order moving average approximation of Lan and Meyer-Gohde (2013c).

6 Estimation using Risky Linear Approximations

I begin by exploring the properties of the likelihood function for the risky linear approximation under controlled settings. In a Monte Carlo experiment, I examine the speed and accuracy of the the likelihood function calculated using the Kalman filter for the risky linear approximation with the likelihood function for the standard linear approximation, likewise calculated with the Kalman filter, and the third-order standard and nonlinear moving average perturbations, where the likelihood function is calculated with the particle filter. I find the risky linear approximation meets or exceeds the computationally intensive nonlinear estimation routines.

I then turn to the Bayesian estimation of risk aversion and the standard deviation of technology growth rate shocks using US post war data on consumption and output growth and the excess return. I find that the posterior of the risky linear approximation calls for higher risk aversion that posited by the prior, whereas the likelihood function for standard linear approximation is entirely flat in the dimension of risk aversion and the likelihood function for the third order perturbations suffer sufficiently from sampling variation to prevent reliable inference.

6.1 Monte Carlo Study of Estimation Properties

Here, I study the ability of the risky linear approximation to estimate nonlinear deep parameters beyond the reach of standard linear approximations and compare the efficiency with which it is able to do such with perturbation-based particle filters that enjoy relative solution efficiency advantages over alternative nonlinear methods—see, e.g., Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006), though Fernández-Villaverde and Rubio-Ramírez (2007) note that perturbation is neither required for nor necessarily the preferred method for taking every model to the particle filter.

For the estimation exercise, I generate two $10,000$ period series of y_t, one for each calibration in section , using a third order nonlinear moving average, see Lan and Meyer-Gohde (2013c). I then estimate parameters one at a time, holding all other parameters constant, using different solution methods. The methods I will compare are the risky linear method of the foregoing sections, standard linearization, a standard state space third order perturbation, and the third order nonlinear moving average used to generate the data. I will present the results for risk aversion, γ, and the standard deviation of growth shocks, $\overline{\sigma}$.

The risky linear method, however, maintains linearity in states and shocks, which, given the assumed normality of growth rate shocks, enables the use of the Kalman filter. Here, I choose the ergodic mean approximation of section 3.1.2 so that the mean of the risky linear approximation coincides with the approximation, in σ, of the ergodic mean of the true nonlinear model. The standard linearization is also estimated with the Kalman filter. The standard approach for the two nonlinear perturbations, nonlinear moving average and standard state space, is to estimate using a sequential importance sampler with resampling, i.e., the particle filter, see Fernández-Villaverde and Rubio-Ramírez (2007), which simulates the entire distributions of the prediction and estimate steps rather than just the first two moments needed in the Gaussian or linear mean-square optimal cases that underly the Kalman filter.Unfortunately, the particle filter can be very demanding computationally, precluding its use currently in many policy relevant models.[33] I set the number of particles in the filter to be $40,000$ and add measurement noise accounting for 1% of the variance of y_t to operationalize a version of the particle filter following, e.g. Bidder and Smith (2012).

[Figure 3 about here.]

In figure 3 the likelihood function, normalized relative to the maximum likelihood value for each method, of risk aversion, γ, and the standard deviation of technology growth shocks, $\overline{\sigma}$, are plotted for the baseline calibration. The standard linear approximation is a certainty equivalent approximation and changes in risk aversion, figure 3a, have no effect on the approximation: the likelihood function is entirely flat in this dimension. The risky linear approximation advocated in previous sections, however, is not certainty equivalent and correctly estimates the level of risk aversion in figure 3a. Both of the particle filter based policy functions correctly estimate the degree of risk aversion, but as can be seen in figure 3a, there is clearly sampling variation and the number of particles would clearly need to be increased past $40,000$ to operationalizes a numerical maximization routine. As can be gathered from the scale of the y axis in figure 3a, though, risk aversion of this small degree is only very weakly identified, placing high demands on the particle filters; the risky linear approximation, however, has no difficulties with this weak identification. All four of the methods correctly estimate the standard deviation of growth shocks, as can be seen in figure 3b. The likelihood cuts for both of the particle filter estimated perturbations coincide and the risky and standard linear approximations display slightly more dispersion than the perturbation methods.

[Figure 4 about here.]

The likelihood cuts, expressed relative to the maximum log likelihood value for each method, of risk aversion, γ, and the standard deviation of technology growth shocks, $\overline{\sigma}$, are plotted in figure

[33] van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012) provides a notable exception, yet due to the demands of the particle filter, they model inflation exogenously and are primarily concerned with the the estimation exercise itself.

4 for the extreme calibration. Again, the standard linear approximation is a certainty equivalent approximation and changes in risk aversion have no effect on the approximation as can be garnered from the entirely flat likelihood function figure 4a. Once again, the risky linear approximation advocated in previous sections, however, is not certainty equivalent and correctly estimates the level of risk aversion, albeit with slightly more dispersion relative to the particle based filters. Both of the particle filter based policy functions correctly estimate the level of risk aversion and nearly coincide in figure 4a. Note that sampling variation in the particle filters is not noticeable in figure 4a, as risk aversion is clearly more strongly identified. Turning to the standard deviation of growth shocks in figure 4b, the standard linear approximation clearly fails to correctly estimate this parameter. As the standard linearization does not capture risk aversion, it attributes the increase in risk sensitivity to an increase in risk itself. The risky linear approximation and the two particle filter based perturbations correctly estimate the standard deviation, with the risky linear approximation and data-generating-process, the nonlinear moving average, coinciding.

[Figure 5 about here.]

Figures 5 and 6 display likelihood cuts under the baseline calibration for output growth—$\log(Y_t) - \log(Y_{t-1})$—and for the ex post risk premium (45). Output growth is an observable series, yet, as figure 5a indicates, this series is unable to reveal the degree of risk aversion. Risk in this model is constant, as opposed to a model with say time varying volatility, and first differencing removes the constant correction for risk in the policy functions, eliminating the role for risk in the observable, as all approximations reflect with their flat likelihood surfaces.

The story is different with data generated by the ex post risk premium. Though not entirely driven by precautionary motives as is the unobservable ex ante risk premium, this measure reveals significant information on the level of risk aversion, as indicated by the curvature of the likelihood function in 6a. With the the effect of increased of risk sensitivity incorporated, all measures but the standard linear approximation agree upon a relative reduction of the source of constant risk, the standard deviation of growth shocks—see figure 6b.

[Figure 6 about here.]

In table 3 the different computation costs, measured in terms of computation time per likelihood evaluation.[34] As can be seen, the the risky linear was negligibly slower than the standard linear with the additional costs coming from the need to calculate the third order perturbation that delivers the higher order derivatives used to correct the linear terms for risk. Compared to the perturbation solutions that use the particle filter, the difference is striking. The risky linear method of the previous sections is four orders of magnitude faster than the particle filter based methods. This despite their similar performance in estimating the parameters and the the choice of the number of particles appears to have been conservative.

[Table 3 about here.]

The estimation exercise provides strong evidence in favor of the method I have presented in the previous sections. It is as successful as, or accounting for potential sampling variability with particle filters more successful than, higher order perturbation methods in identifying nonlinear parameters, like risk aversion, that standard linear approximations cannot identify while maintaining the computational efficiency provided by the linear in state and shock framework. I find that the the risky linear approach advocated in the foregoing sections clearly dominates the more computationally intensive particle-filter based methods, while performing favorably in terms of accuracy and efficiency of the estimation. For comparison, the standard linear approximation fails entirely at estimating risk aversion.

[34] Comparisons computed on an Intel Xeon E5-2690 with 16 cores at 2.90 GHz on Matlab R2013b. Approximately 61% of the processor resources were used by the particle filter at any given point in time during the calculations.

6.2 US Post-War Estimation of Risk and Risk Sensitivity

I now turn to the estimation of risk and risk sensitivity using post war US data. While estimating, I take a Bayesian perspective following standard practice in the DSGE literature.[35] Taking the results of the previous section into account, I shall include ex post risk premium along with consumption and output growth in the data set.[36] I find that the risky linear approximation introduced here appeals for more risk and risk aversion as one would expect with the inclusion of the ex post risk premium. The particle filter based methods suffer from sampling variation close to the posterior mode, which makes estimation with my relatively flat prior infeasible.

[Table 4 about here.]

Table 4 contains the priors of the standard deviation of growth shocks and risk aversion. Both priors are relatively loose, with the prior on risk aversion centered roughly in between the two values of the calibrated model. The standard deviation of the growth shock has its prior mean and mode below the calibrated values but assigns substantial probability mass to the region around that value. Table 4 contains point estimates from the posterior from the risky and standard linear approximations. The risky linear approximation favors more risk aversion and more risk than the standard linear approximation, whose estimate of risk aversion is entirely prior driven with prior and posterior modes coinciding and the likelihood function entirely flat along this dimension.

[Figure 7 about here.]

Figure 7 depicts the posterior as well as the likelihood using the risky linear approximation. The likelihood function, figure 7b, indicates that the data is informative in both dimensions using the risky linear method of this paper. While the likelihood and posterior, figure 7a, both favor a similar value for the standard deviation of growth rate shocks, $\overline{\sigma}$, they differ substantially over

[35] See Smets and Wouters (2003) and Smets and Wouters (2007) for prominent and Del Negro, Schorfheide, Smets, and Wouters (2007) and An and Schorfheide (2007) for instructive examples of Bayesian estimation of DSGE models.

[36] See appendix A.8 for details on the data series.

the parameter controlling risk aversion, γ. As discussed also by, e.g., Tallarini (2000), production models with recursive utility can match the slope of the market line (or market price of risk) but require exorbitant levels of risk aversion to come close to the average risk premium, see table 5. The posterior tempers this tendency, yielding a modest outward shift in the distribution of risk aversion relative to the prior.

[Figure 8 about here.]

In figure 7, the posterior and likelihood using the standard linear approximation can be found. As was to be expected from the results of the preceding sections, the likelihood is flat along the dimension of the parameter controlling risk aversion. In other words, the precautionary component of the risk premium in the data is entirely ignored and risk aversion is completely prior driven.

The posteriors and likelihoods for the nonlinear moving average perturbation can be found in figure 9.[37] As was the case for two of the four sets of synthetic data from the calibrated models, sampling variation in the particle filter is visible here with the post war US data set with the dimension in the risk aversion parameter, γ, being most obviously impacted.[38] This is not surprising, as the likelihood surface for the risky linear approximation indicates that this dimension of the likelihood function is nearly flat, especially for values of the standard deviation of growth shocks, $\overline{\sigma}$, close to the mode. Nonetheless, for larger values of $\overline{\sigma}$, a clear upward slope for larger values of γ emerges, consistent with the model requiring more risk aversion to increase the risk premium.

[Figure 9 about here.]

Table 5 gives the asset pricing variable moments.[39] As discussed above, the model does not match the magnitude of the empirical risk premium. The risky linear approximation is, however,

[37]The results for the standard perturbation were essentially the same and have been omitted for brevity.

[38]For the figures here, I increased the number of particles to 100,000, which improved but as is clear from the figure did not eliminate the sampling variation of the particle filters.

[39]The macroeconomic variables remain essentially unchanged as on the risk aversion has been changed substantially and it is known, see Tallarini (2000) for example, that macroeconomic variables are virtually invariant to the level of risk aversion, holding the intertemporal elasticity of substitution constant. Tables with empirical as well as the posterior model based business cycle measures can be found in appendix A.9.

able to bring the market price of risk from the pricing kernel ($\text{std}(m_t)/E[m_t]$) and the Sharpe ratio from the excess return on risky capital ($E[r_t^k - r^f]/\text{std}(r_t^k - r^f)$) close to the empirical market price of risk as measured by the NYSE value weighted portfolio over the secondary market rate for three month Treasury bill.[40] As the standard linear approximation does not generate a risk premium at all, its Sharpe ratio is zero, and the standard linear approximation produces a market price of risk that is half the size as generated by the risky linear approximation.

[Table 5 about here.]

Informing the estimation with the empirical risk premium along with consumption and output growth leads the posterior with the risky linear approximation to favor a higher level of risk aversion than under the prior. The standard linear approximation is invariant to the level of risk aversion and so the likelihood function is unable to inform the posterior. As the likelihood function is rather flat in the dimension of risk aversion, full nonlinear estimation is infeasible as the particle filter suffers from a sampling variation largen enough to mask the curvature in the likelihood function. Under the posterior mode estimates, the risky linear approximation is unable to match the empirical risk premium but does bring the model's predictions of the market price of risk and the Sharpe ratio closer to the observed market price of risk for the NYSE value weighted portfolio.

7 Conclusion

I have derived and analyzed a risky linear approximation of the policy function for DSGE models. The method solves linear equations in standard perturbation output, requiring neither fix point nor other recursive methods. This direct approach along with the minimal costs associated with standard perturbation methods allow me provide a certainty non equivalent method suitable for the estimation and analysis of policy relevant DSGE policy under risk without needed to turn to the particle filter or to trust the unknown convergence properties of alternate algorithms purporting to correct for risk.

[40] A description of the post war US data used for the empirical values can be found in appendix A.8

Finally, the method presented is the first, to my knowledge, to construct a linear approximation around an approximation of the ergodic mean of the underlying nonlinear model, due to the choice of the method of operating implicitly with the unknown policy function instead of the equilibrium conditions of the model as is done in other, competing methods.

In the chosen application, a risk-sensitive real business cycle model with long run risk, I find that the risky linear approximation is a uniform improvement over the standard linear approximation and as risk becomes more important in the the model, the accuracy of the algorithm becomes comparable to second and third perturbations in the vicinity of the stochastic steady state despite its linearity in states and shocks. The method produces impulse responses identical to those generated by a third order nonlinear moving average, able to model the responses of conditional asset pricing variables to shocks, which are beyond the reach of standard linear approximations. Finally, in a estimation exercise, I show that the risky linear approximation estimated using the Kalman filter correctly identifies risk and risk aversion (here, different from the inverse elasticity of intertemporal substitution due to the recursive utility formulation) along with the particle filter estimations of standard perturbation and nonlinear moving average approximation. Thus, the risky linear approximation combines the efficiency in estimation (with the Kalman filter here being four orders of magnitude faster than the particle filter) of linear formulations with the information from nonlinear approximations need to identify parameters such as the degree of risk aversion that are beyond the reach of standard linear approximations. Indeed, in the application to post war US data, the likelihood function is entirely flat in the dimension of risk aversion for a standard linear approximation and sufficiently flat for third order perturbations using the particle filter that sampling variation precludes reliable inference.

The method here could be extended using the change of variable transformations studied by Fernández-Villaverde and Rubio-Ramírez (2006) to maximized the accuracy of the risky linear approximation. Likewise conditional linear approximations as applied by Justiniano and Primiceri (2008) to study volatility shifts in post war US data could be risk adjusted to capture the precautionary effects neglected in their work. Finally, the method developed here could be applied to policy relevant models that require capturing risk, e.g., to match financial market date, but whose size precludes the application of alternative nonlinear methods, e.g., the computational costs of the particle filter are too burdensome. In work in progress, Kliem and Meyer-Gohde (2013) apply the risky linear method developed here to estimate and analyze the effects of monetary policy in a medium scale DSGE macro finance model of the nominal term structure.

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A Appendix

A.1 Detrended Model

Detrending with $x_t \doteq X_t / e^{Z_t}$ ($u_t \doteq U_t / e^{\nu Z_t}$) gives

(A-1) $$u_t = \left[(1-\beta) \left(c_t^{\nu} (1-L_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[(u_{t+1} e^{\nu a_t})^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}}$$

(A-2) $$c_t + k_t = e^{-\xi a_t} k_{t-1}^{\xi} L_t^{1-\xi} + (1-\delta) e^{-a_t} k_{t-1}$$

(A-3) $$1 = E_t \left[M_{t+1} \left(\xi e^{(1-\xi) a_{t+1}} k_t^{\xi-1} L_{t+1}^{1-\xi} + 1 - \delta \right) \right]$$

(A-4) $$M_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} e^{a_{t+1}} \right)^{\nu \frac{1-\gamma}{\theta} - 1} \left(\frac{1-L_{t+1}}{1-L_t} \right)^{\frac{(1-\nu)(1-\gamma)}{\theta}} \left(\frac{(u_{t+1} e^{\nu a_{t+1}})^{1-\gamma}}{E_t \left[(u_{t+1} e^{\nu a_{t+1}})^{1-\gamma} \right]} \right)^{1-\frac{1}{\theta}}$$

(A-5) $$\frac{1-\nu}{\nu} \frac{c_t}{1-L_t} = (1-\xi) e^{-\xi a_t} k_{t-1}^{\xi} L_t^{-\xi}$$

A.2 Matrix Derivatives

That is

(A-6) $$g_{z^j \sigma^i} \doteq \mathscr{D}^{j+i}_{z_{t-1}^j \sigma^i} \{ y(\sigma, z_t) \} \doteq \left(\begin{bmatrix} \frac{\partial}{\partial z_{1,t-1}} & \cdots & \frac{\partial}{\partial z_{n_z,t-1}} \end{bmatrix}^{\otimes [j]} \otimes \left(\frac{\partial}{\partial \sigma} \right)^{\otimes [i]} \right) \otimes y_t$$

(A-7) $$= \left(\begin{bmatrix} \frac{\partial}{\partial z_{1,t-1}} & \cdots & \frac{\partial}{\partial z_{n_z,t-1}} \end{bmatrix}^{\otimes [j]} \left(\frac{\partial}{\partial \sigma} \right)^{i} \right) \otimes y_t$$

where the second line follows as σ is a scalar. The terms $\left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i \right]$ in (4) collect all the coefficients associated with the j'th fold Kronecker product of the state vector, $(z_t - \bar{z})$. Higher orders of σ in $g_{z^j \sigma^i}$ correct the Taylor series coefficients for uncertainty by successively opening the coefficients to higher moments in the distribution of future shocks.[41] I will take the availability of these standard perturbation Taylor series as given.[42]

With f and y both being vector-valued functions that take vectors as arguments, their partial derivatives form hypercubes. I use the method of Lan and Meyer-Gohde (2013c) that differentiates conformably with the Kronecker product, allowing us to maintain standard linear algebraic struc-

[41] A similar interpretation can be found in Judd and Mertens (2012) for univariate expansions and in Lan and Meyer-Gohde (2013c) for expansions in infinite sequences of innovations.

[42] See Jin and Judd (2002) and Lan and Meyer-Gohde (Forthcoming) for more on whether these coefficients can indeed be recovered by standard DSGE perturbations.

tures to derive our results.

Definition A.1. *Matrix Derivatives*

Let $A(B):\mathbb{R}^{s\times 1}\rightarrow\mathbb{R}^{p\times q}$ *be a matrix-valued function that maps an* $s\times 1$ *vector* B *into an* $p\times q$ *matrix* $A(B)$*, the derivative structure of* $A(B)$ *with respect to* B *is defined as*

$$A_B \doteq \mathscr{D}_B\{A\} \doteq \left[\frac{\partial}{\partial b_1} \quad \dots \quad \frac{\partial}{\partial b_s}\right]\otimes A \tag{A-8}$$

where b_i *denotes i'th row of vector B; n'th derivatives are*

$$A_{B^n} \doteq \mathscr{D}^n_{(B)^n}\{A\} \doteq \left(\left[\frac{\partial}{\partial b_1} \quad \dots \quad \frac{\partial}{\partial b_s}\right]^{\otimes[n]}\right)\otimes A \tag{A-9}$$

Details of the associated calculus that generalizes familiar chain and product rules as well as Taylor approximations to multidimensional settings can be found in Lan and Meyer-Gohde (2013c) and Lan and Meyer-Gohde (Forthcoming).

A.3 Proof of Proposition 3.1

Successive differentiation of (14) yields equations recursively linear in $\overline{y}_{\sigma^i}$ taking as given lower order terms of the form $\overline{y}_{\sigma^i}$ and derivatives of g with respect to y_{t-1} and σ. For solvability, following the implicit function theorem, the matrix g_y, the first derivative of the policy function at the deterministic steady state with respect to endogenous variables, must have all eigenvalues inside the unit circle; this holds under local saddle stability of (1).

A.4 Proof of Corollary 3.2

For a second-order (in σ) approximation of the stochastic steady state, differentiate $\overline{y}(\sigma)=g(\overline{y}(\sigma),0,\sigma)$ at $\sigma=0$ once for

$$\overline{y}'(0)=g_y\overline{y}'(0)+g_\sigma=\left(I-g_y\right)^{-1}g_\sigma=0 \tag{A-10}$$

and twice for

$$\overline{y}''(0)=g_{y^2}\overline{y}'(0)^{\otimes[2]}+2g_{y\sigma}\overline{y}_\sigma+g_y\overline{y}''(0)+g_{\sigma^2}=\left(I-g_y\right)^{-1}g_{\sigma^2} \tag{A-11}$$

Thus, up to second order in σ, the stochastic steady state is

$$\overline{y}^{\text{stoch}} \approx \overline{y} + \frac{1}{2}\left(I - g_y\right)^{-1} g_{\sigma^2} \tag{A-12}$$

as claimed in corollary 3.2.

A.5 Proof of Proposition 3.3

Successive differentiation of (2) with respect to σ evaluated at the deterministic steady state gives recursive equations $\mathscr{D}_{\sigma^i} y_t$ that depend on lower order derivatives of $\mathscr{D}_{\sigma^i} y_t$,[43] derivatives of the g function evaluated at the deterministic steady state, and the exogenous vector ε_t. Successive differentiation of (18) yields equations recursively linear in $\overline{y}_{\sigma^i}$ taking as given lower order terms of the form $\overline{y}_{\sigma^i}$, derivatives of the g function evaluated at the deterministic steady state, and expectations of terms involving $\mathscr{D}_{\sigma^i} y_t$ and ε_t. For solvability both of the expectations of $\mathscr{D}_{\sigma^i} y_t$ and of derivatives of (18), following the implicit function theorem, the matrix g_y, the first derivative of the policy function at the deterministic steady state with respect to endogenous variables, must have all eigenvalues inside the unit circle; this holds under local saddle stability of (1). Under this condition and if the moments of ε_t exists and are finite, the terms involving expectations and the derivatives of (18), $\overline{y}_{\sigma^i}$, can be solved uniquely from the given moments of ε_t and derivative information of the g function evaluated at the deterministic steady state.

A.6 Proof of Corollary 3.4

For a second-order (in σ) approximation of the ergodic mean, differentiate $\overline{y}(\sigma) = E\left[g(y_{t-1}, \sigma\varepsilon_t, \sigma)\right]$ at $y_{t-1} = \overline{y}(0)$ and $\sigma = 0$ once for

$$\overline{y}'(0) = E\left[g_y \mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon \varepsilon_t + g_\sigma\right] = \left(I - g_y\right)^{-1} g_\sigma = 0 \tag{A-13}$$

and twice for

$$\overline{y}''(0) = E\left[g_y \mathscr{D}_{\sigma^2}\{y_{t-1}\} + g_{y^2} \mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]} + 2 g_{y\varepsilon} \varepsilon_t \otimes \mathscr{D}_\sigma\{y_{t-1}\}\right. \tag{A-14}$$

[43] $\mathscr{D}_{\sigma^i} y_t$ denotes the i'th order derivative of y_t with respect to σ. The alternative notation, y_{σ^i}, refers to the i'th derivative of y_t with respect to its third argument, i.e., the "direct" derivative with respect to σ, neglecting derivatives involving σ that enter through the term $\sigma\varepsilon_t$ that are included in the notation $\mathscr{D}_{\sigma^i} y_t$.

(A-15) $$+2g_{y\sigma}\mathscr{D}_\sigma\{y_{t-1}\} + 2g_{\varepsilon\sigma}\varepsilon_t + g_{\varepsilon^2}\varepsilon_t^{\otimes[2]} + g_{\sigma^2}\Big]$$

(A-16) $$= \left(I - g_y\right)^{-1}\left(g_{y^2}E\left[\mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]}\right] + g_{\varepsilon^2}E\left[\varepsilon_t^{\otimes[2]}\right] + g_{\sigma^2}\right)$$

(A-17) $$= \left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right)E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

where the last line follows from $E\left[\mathscr{D}_\sigma\{y_t\}^{\otimes[2]}\right] = E\left[\left(g_y\mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon\varepsilon_t + g_\sigma\right)^{\otimes[2]}\right] = g_y^{\otimes[2]}E\left[\mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]}\right] + g_\varepsilon^{\otimes[2]}E\left[\varepsilon_t^{\otimes[2]}\right]$ Thus, up to second order in σ, the ($\sigma = 1$) ergodic mean is

$$\overline{y}^{\text{mean}} \approx \overline{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right)E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

as claimed in corollary 3.4.

A.7 Proof of Corollary 3.7

For a second-order (in σ) approximation of $\overline{y_y}(\sigma)$, differentiate $\overline{y_y}(\sigma) = g_y(\overline{y}(\sigma), 0, \sigma)$ once for

(A-18) $$\overline{y_y}'(0) = g_{y^2}\overline{y}'(0) \otimes I_{n_y} + g_{\sigma y} = 0$$

and twice for

(A-19) $$\overline{y_y}''(0) = g_{y^2}\left(\overline{y}''(0) \otimes I_{n_y}\right) + g_{\sigma^2 y}$$

Thus, up to second order in σ, the ($\sigma = 1$) derivative in y is

$$\overline{y_y}(1) \approx g_y + \frac{1}{2}\left(g_{y^2}\left(\overline{y}''(0) \otimes I_{n_y}\right) + g_{\sigma^2 y}\right)$$

as was claimed in corollary 3.7. Analogous derivations follow for $\overline{y_\varepsilon}(\sigma) \approx \overline{y_\varepsilon}(0) + \overline{y_\varepsilon}'(0)\sigma + \frac{1}{2}\overline{y_\varepsilon}''(0)\sigma^2$.

A.8 Data

I use post-war US macroeconomic and asset priucing data to calibrate the model in section 5 and to estimate risk aversion and the standard deviation of technology growth shocks in section 6.

All series are quarterly. Investment is defined as the sum of the National Income and Product Accounts (NIPA) measures of Personal Consumption Expenditures on Durable Goods, Private Nonresidential Fixed Investment, and Private Residential Fixed Investment; Consumption as the sum of the NIPA measures of Personal Consumption Expenditures on Nondurable Goods and Services; Output is Gross Domestic Product (GDP) expressed at an annual rate; Hours are measured by Hours

Worked by Full-Time and Part-Time Employees, interpolated to a quarterly series by the growth rate of Civilian Noninstitutional Population series. Investment, Consumption, and Output are expressed in real per capita terms by deflating by the Civilian Noninstitutional Population series and the chain-type GDP deflator.

The risky return is the return on the NYSE value weighted portfolio from the CRSP dataset and the risk-free return is secondary market rate for the three month Treasury bill. Both returns have been deflated by the implicit deflator of the Personal Consumption Expenditures Nondurables and Services series.

A.9 Business Cycle Tables

Table 6 summarizes the first two moments of output, consumption, investment, and hours.

[Table 6 about here.]

Table 7 summarizes the first two moments of output, consumption, investment, and hours from the model of section 4 evaluated at the posterior mode with the risky linear approximation.

[Table 7 about here.]

Table 1: Common Calibration

Parameter	$\overline{a}$	δ	ν	β	ξ
Value	0.46%	0.0196	0.357	0.991	0.3

$\overline{a}$—1948:3-2013:2 average output growth
Remaining values from Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012)

Table 2: Baseline and Extreme Calibration

	Baseline			Extreme		
Parameter	γ	ψ	$\overline{\sigma}$	γ	ψ	$\overline{\sigma}$
Value	5	1.008	1.12625%	40	1.0085	1.1269%

γ and η—Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao's (2012) baseline and extreme values
$\overline{\sigma}$ and ψ—1948:3-2013:2 average output and consumption growth volatilities

Table 3: Computational Costs: Monte Carlo Estimation

Method	Linear	Risky Linear	3rd Order Pert.	3rd Order Pert. (pruned)
Evaluation Time	0.44	0.47	430	690

in seconds, per likelihood evaluation

Table 4: Priors and Posteriors

	γ	$\overline{\sigma}$
	Priors	
Type	Shifted Gamma	Inverse Gamma
Mean	20	0.22%
Mode	14.737	0.11%
Standard Deviation	10	0.6%
Domain	$(1,\infty)$	$(0,\infty)$
	Posteriors	
Risky Linear Mode	29.296	1.0032%
Standard Linear Mode	14.737	0.9911%

Table 5: Asset Return Properties

	Empirical		Risky Linear		Standard Linear	
Return	Mean	Std. Dev.	Mean	Std. Dev.		
r^k	2.14	8.25	0.5003	0.0801	0.5502	0.0758
r^f	0.26	0.62	0.4980	0.0767	0.5502	0.0726
$r^k - r^f$	1.88	8.25	0.0023	0.0212	0.000	0.0217
Market Price of Risk	0.2283		0.2004		0.1049	
Sharpe Ratio			0.1072		0.0000	

All returns are measured as real quarterly percentage returns.
A description of the post war US data used for the empirical values can be found in appendix A.8.
The model based numbers were derived using the posterior mode from the risky linear model, see table 4

Table 6: Business Cycle Data, 1948:2-2013:2

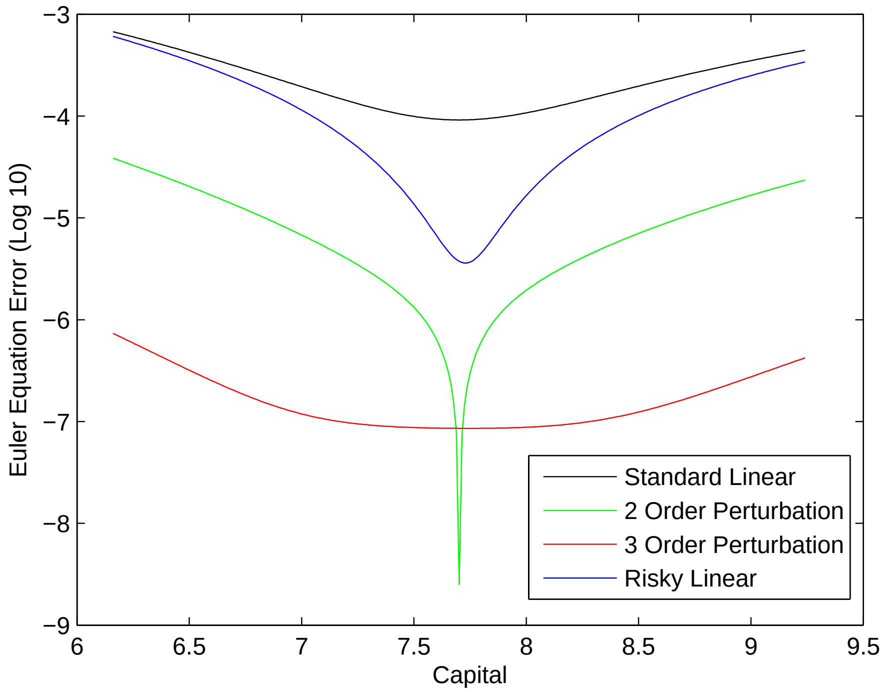
Variable	Mean	Std. Dev.	Relative Std. Dev.	Autocorrelations 1	2	3	Cross Corr. w$\Delta \ln Y_t$
$\Delta \ln Y_t$	0.458	0.988	1.000	0.381	0.266	0.046	1.000
$\Delta \ln C_t$	0.497	0.565	0.572	0.257	0.205	0.074	0.531
$\Delta \ln I_t$	0.420	2.527	2.558	0.335	0.249	0.043	0.662
$\Delta \ln N_t$	0.328	1.188	1.202	-0.020	-0.010	-0.008	0.388
$\ln N_t$	119.993	2.786	2.820	0.999	0.998	0.997	-0.141
$\ln C_t - \ln Y_t$	—	5.956	6.029	0.990	0.979	0.965	-0.173
$\ln I_t - \ln Y_t$	—	7.328	7.418	0.962	0.911	0.843	0.129

All data was retrieved from the Federal Reserve Economic Data (FRED) database of the Federal Reserve Bank of St. Louis.

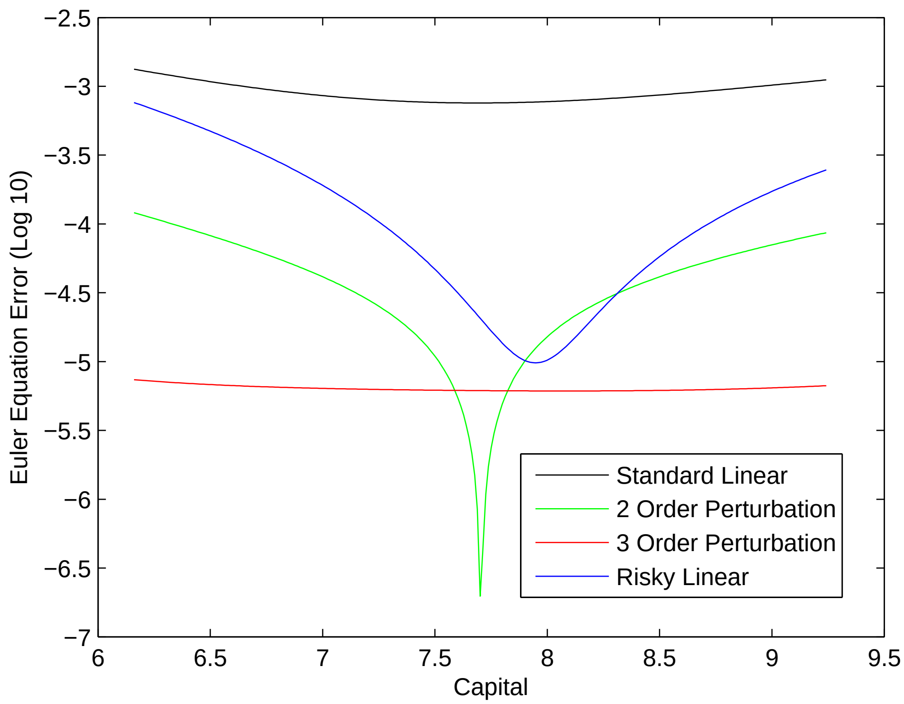
Table 7: Business Cycle Properties, Posterior Mode

Variable	Mean	Std. Dev.	Relative Std. Dev.	Autocorrelations 1	2	3	Cross Corr. w$\Delta \ln Y_t$
$\Delta \ln Y_t$	0.46	0.863	1.000	0.0083	0.0080	0.0078	1.000
$\Delta \ln C_t$	0.46	0.515	0.596	0.0665	0.0641	0.0619	0.992
$\Delta \ln I_t$	0.46	1.712	1.983	-0.0156	-0.0150	-0.0145	0.996
$\Delta \ln N_t$	0	0.231	0.268	-0.0237	-0.0229	-0.0221	0.984
$\ln N_t$	-1.035	0.864	1.001	0.9643	0.9303	0.8980	0.308
$\ln C_t - \ln Y_t$	—	1.341	1.553	0.9643	0.9303	0.8980	-0.308
$\ln I_t - \ln Y_t$	—	3.203	3.710	0.9643	0.9303	0.8980	0.308

Compare with the empirical moments in table 6.

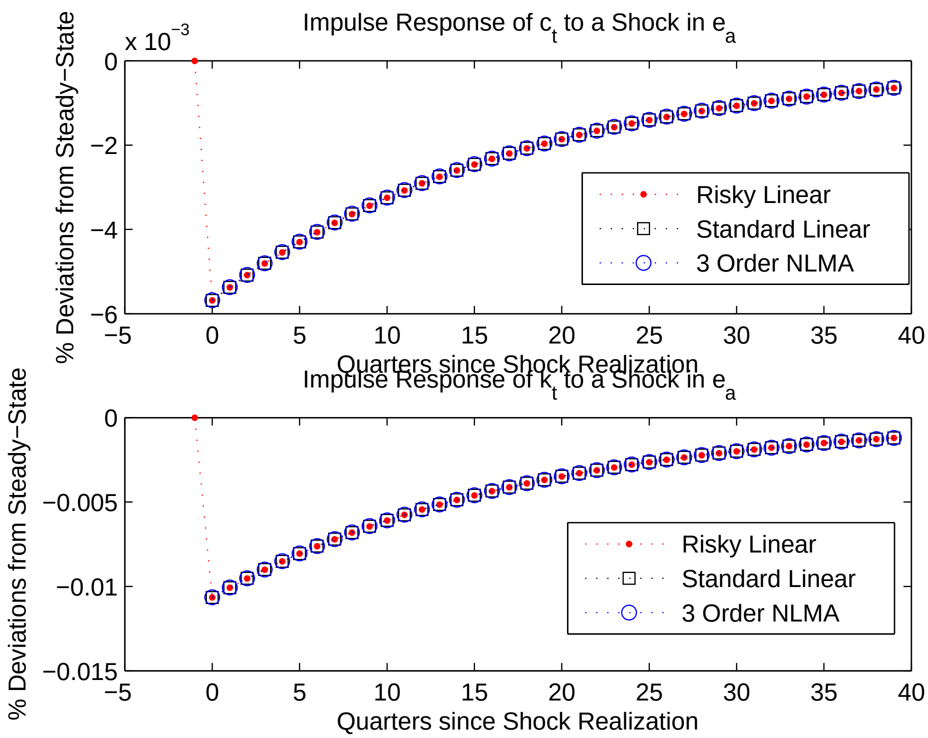


(a) Baseline Calibration

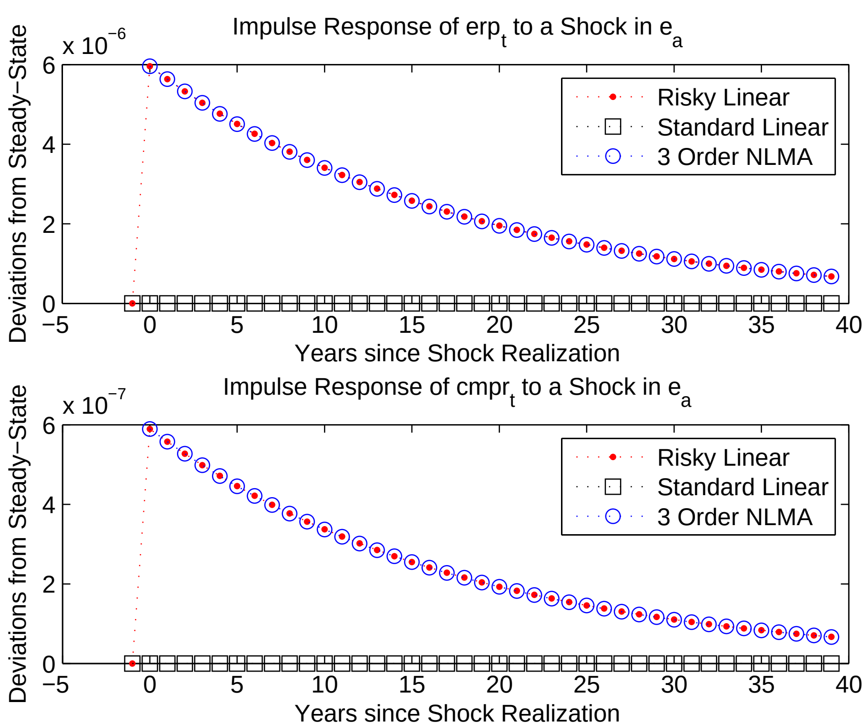


(b) Extreme Calibration

Figure 1: Euler Equation Errors

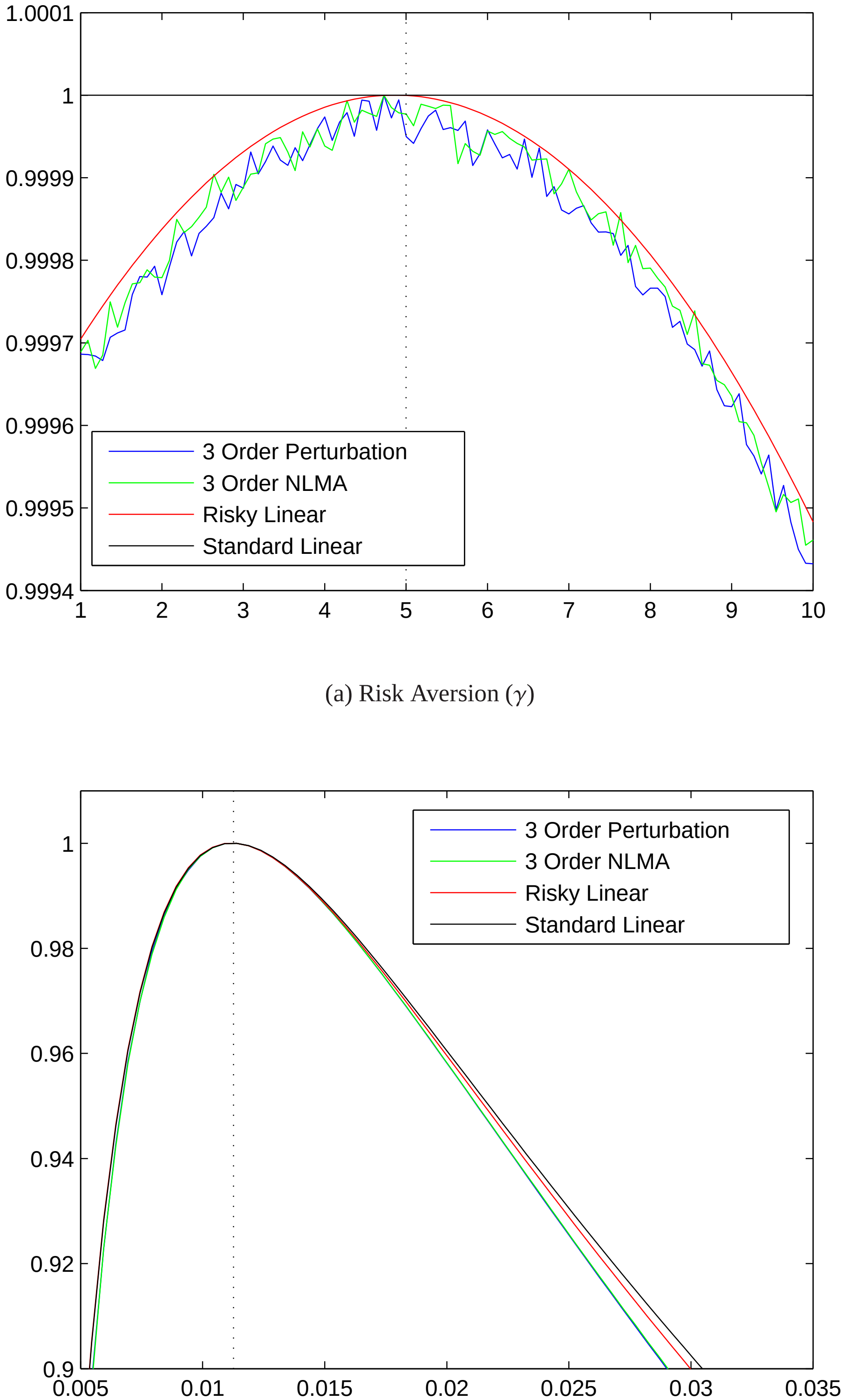


(a) Macroeconomic Variables



(b) Asset Pricing Variables

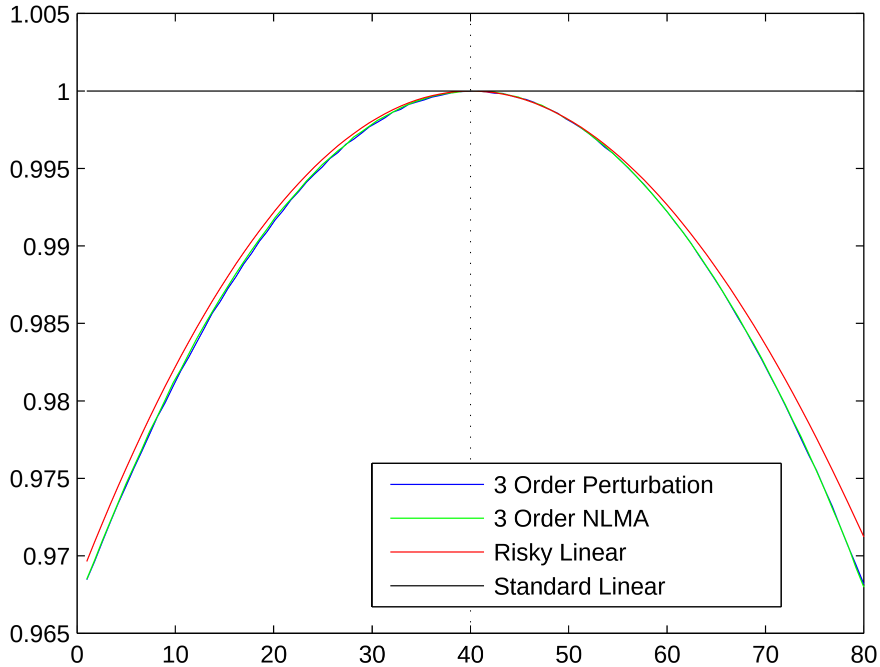
Figure 2: Impulse Response Functions



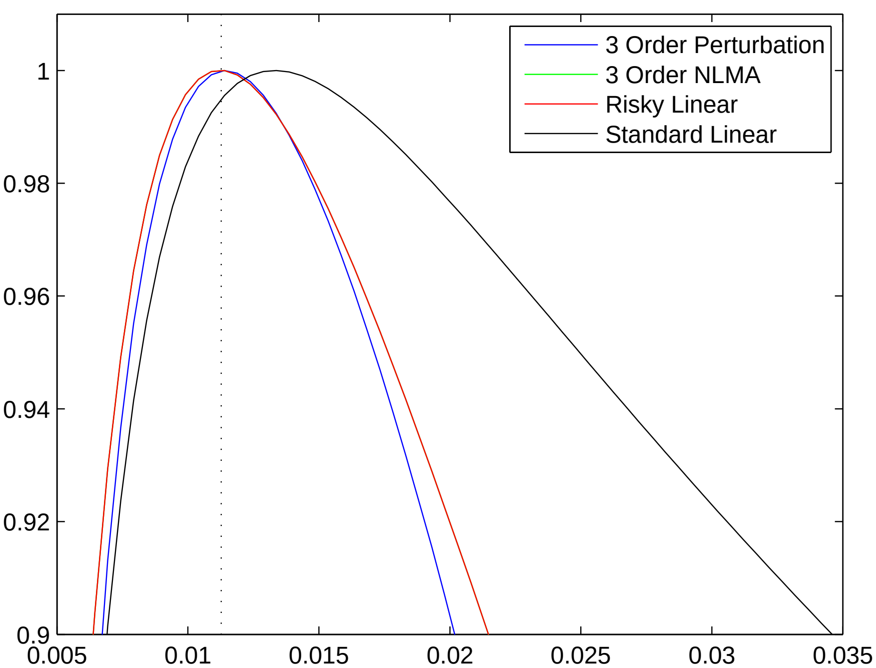
(a) Risk Aversion (γ)

(b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 3: Likelihood Cuts: Baseline Calibration

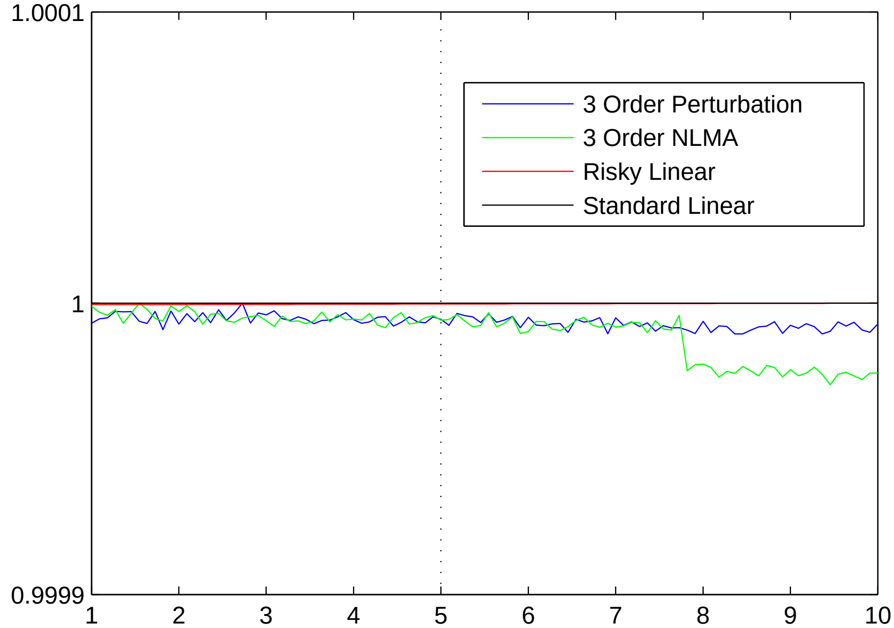


(a) Risk Aversion (γ)

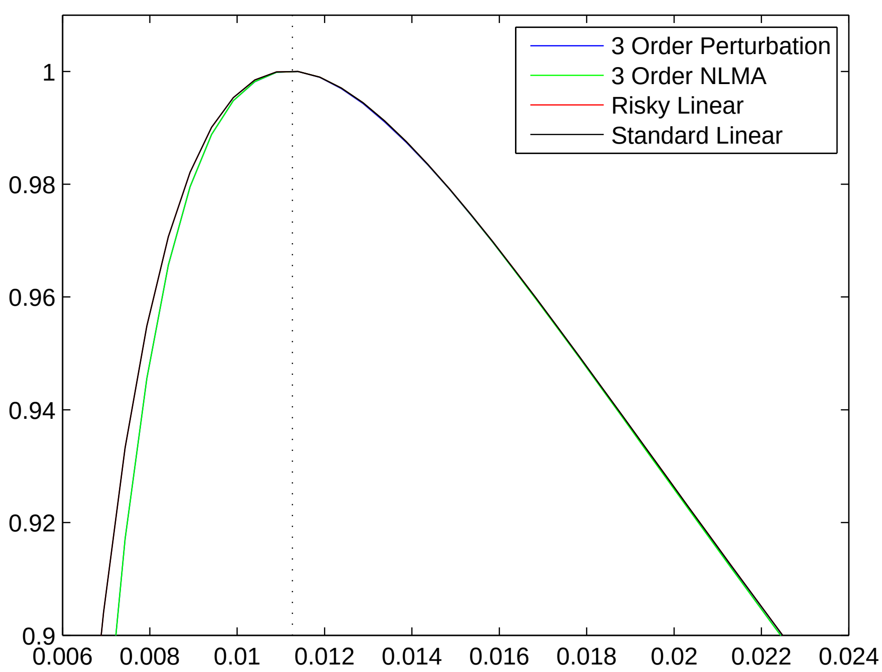


(b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 4: Likelihood Cuts: Extreme Calibration

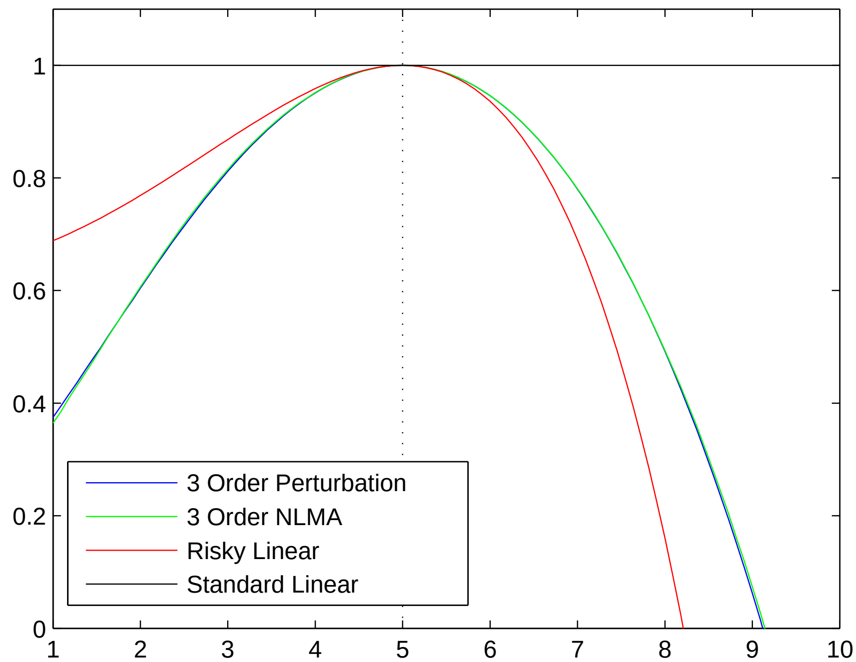


(a) Risk Aversion (γ)

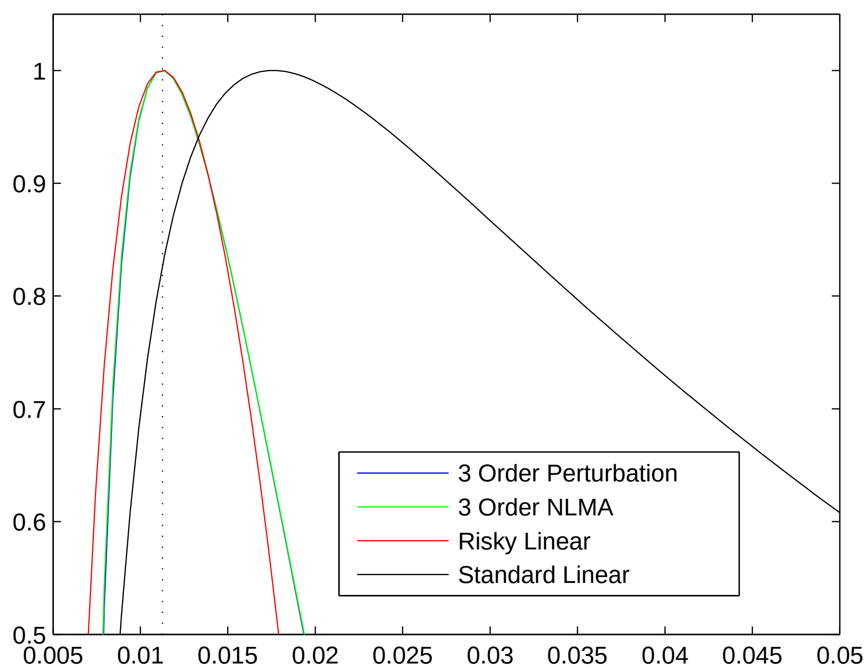


(b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 5: Likelihood Cuts: Baseline Calibration, Data on ΔY_t

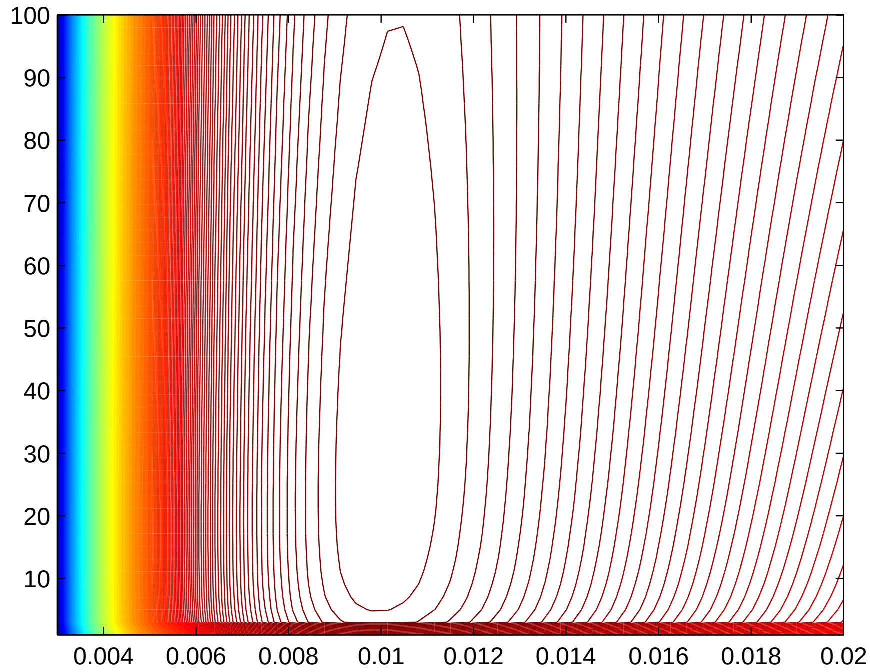


(a) Risk Aversion (γ)

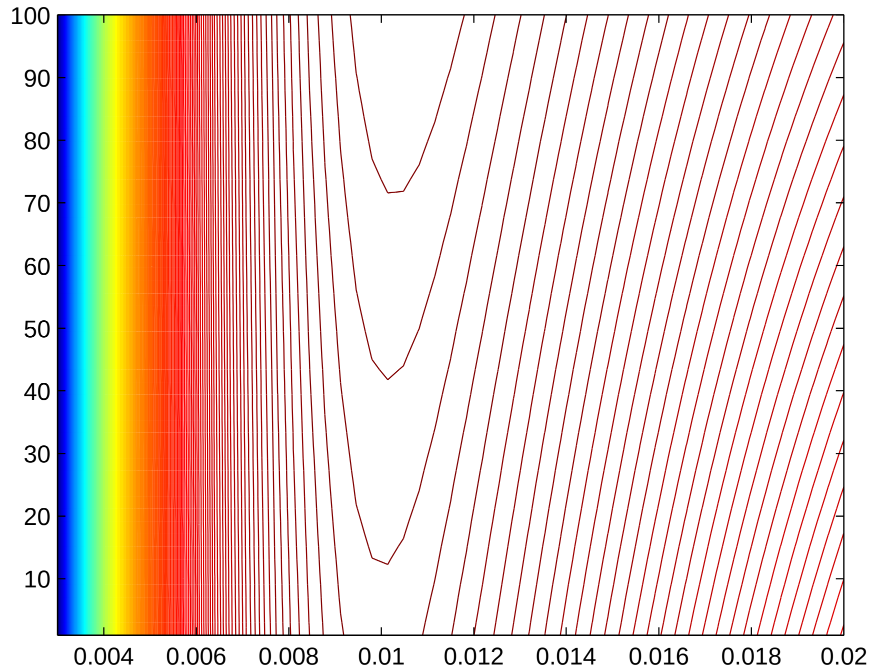


(b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 6: Likelihood Cuts: Baseline Calibration, Data on rp_t

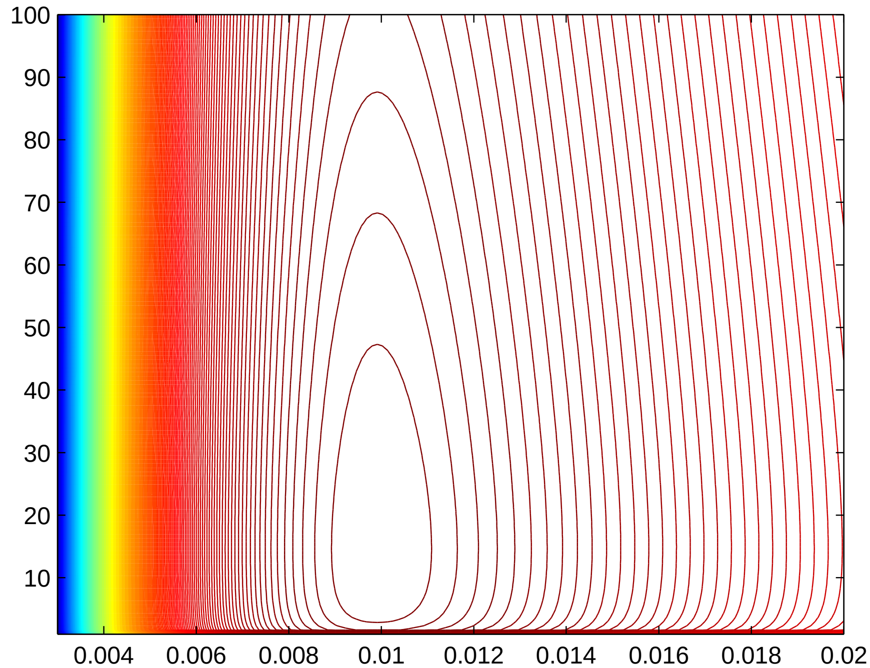


(a) Posterior

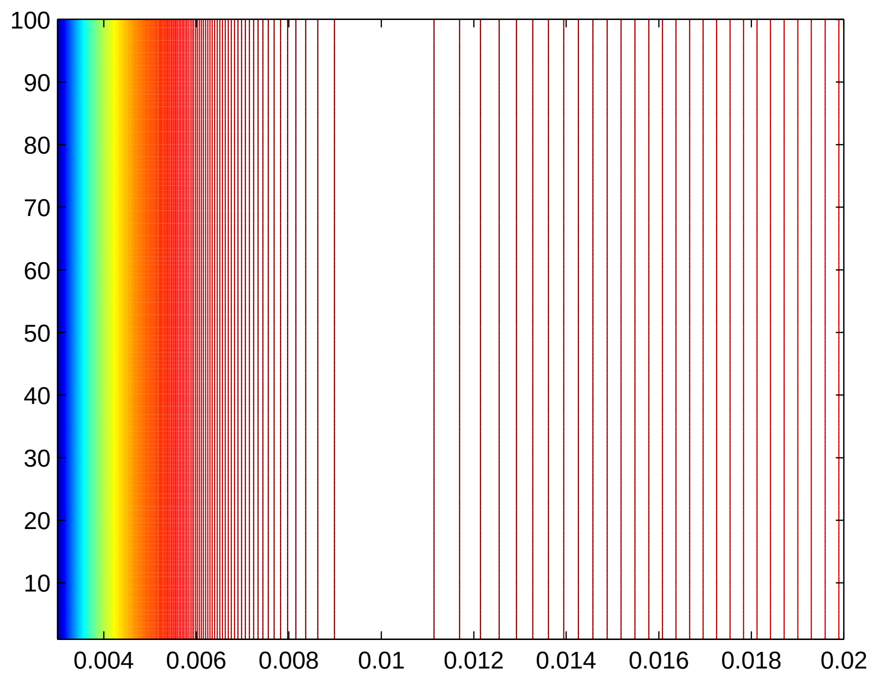


(b) Likelihood

Figure 7: Risky Linear Estimation Results
x-axis: $\overline{\sigma}$; y-axis: γ

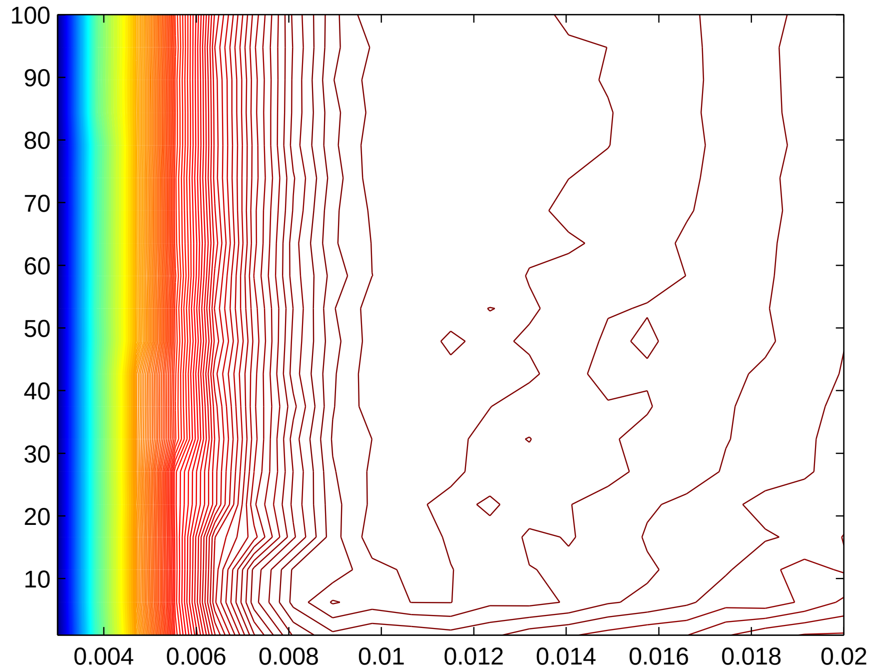


(a) Posterior

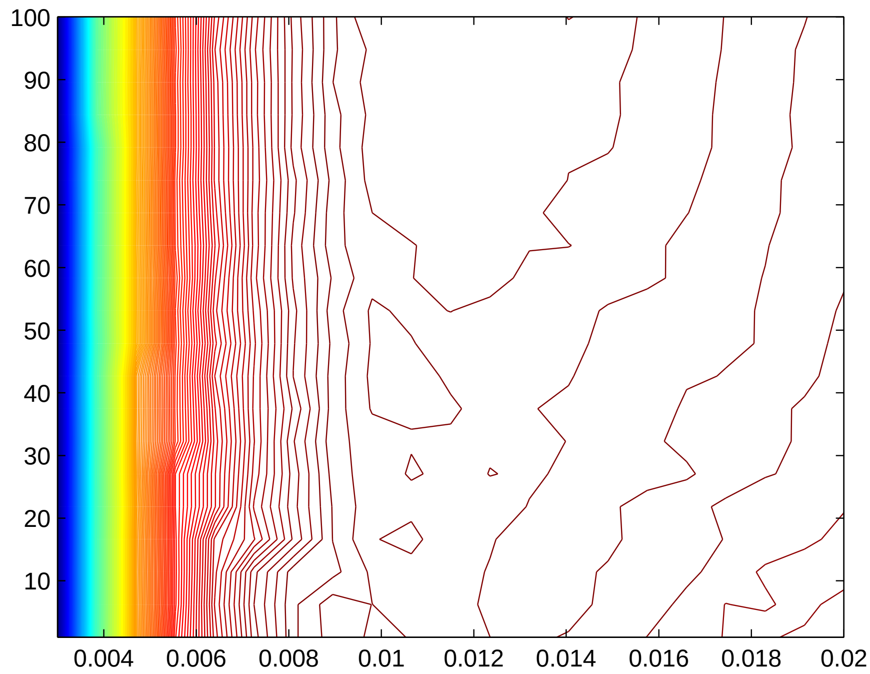


(b) Likelihood

Figure 8: Standard Linear Estimation Results
x-axis: $\overline{\sigma}$; y-axis: γ



(a) Posterior



(b) Likelihood

Figure 9: Third Order Nonlinear Moving Average Estimation Results
x-axis: $\overline{\sigma}$; y-axis: γ